Principal bundles in NC Riemannian geometry

Glavni svežnjevi u nekomutativnoj Riemannovoj geometriji

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Shameless self-promotion

References

B. Ć. and B. Mesland, *Gauge theory on noncommutative Riemannian principal bundles*, `arXiv:1912.04179`

B. Ć., *Non-trivial gauge theory on cleft quantum principal bundles*, (in preparation)

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Note

Today, we specialise to unital NC principal $\mathrm{U}(1)$-bundles with totally geodesic orbits of unit length.

Cod-Hegelian dialectic

Let G be a compact connected Lie group.

What is a NC (Yang–Mills) gauge theory with structure group G?

Thesis (Brzeziński–Majid et al.)

Connections on principal $\mathcal{O}(G)$-comodule algebras.

Antithesis (Chamseddine–Connes et al.)

The spectral action principle on suitable spectral triples.

Synthesis? (cf. Brain–Mesland–Van Suijlekom)

The very latest in unbounded KK-theory.

Basic setup

Let $G = U(1)$, so that $\mathfrak{g} = \mathbb{R}\frac{\partial}{\partial t}$, where

$$\forall f \in \mathcal{O}(G),\ \forall z \in G, \quad (\tfrac{\partial}{\partial t} f)(z) := \lim_{s \to 0} \frac{f(ze^{is}) - f(z)}{s};$$

hence, $\mathfrak{g}^* = \mathbb{R}dt$ for $dt := -i\frac{dz}{z}$ with $(dt, \frac{\partial}{\partial t}) = 1$.

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Thus, a unital G-C^*-algebra (A, α) is *principal* iff

$$\forall n \in \mathbb{Z}, \quad \overline{A_n^* \cdot A_n} = A^G, \quad A_n := \{a \in A \mid \forall z \in G,\ \alpha_z(a) = z^n a\},$$

in which case $A \hookleftarrow A^G$ is a NC topological principal G-bundle.

Example (Matsumoto, cf. Brzeziński–Sitarz)

The θ-deformed $\mathbb{C}$-Hopf fibration $C(S^3_\theta) \hookleftarrow C(S^3_\theta)^G \cong C(S^2)$.

Running example

Fix $\theta \in \mathbb{R}$, so $\mathbb{Z}$ acts on $C(G)$ by $(f \triangleleft_\theta m)(z) \coloneqq f(z \cdot e^{2\pi i m \theta})$.

The *rotation algebra* $A_\theta \coloneqq C(G) \rtimes_\theta \mathbb{Z}$ admits:

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1. an action α of $G \coloneqq U(1) = \widehat{\mathbb{Z}}$ defined by

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2. a G-invariant faithful trace τ defined by
$$\forall m \in \mathbb{Z}, \forall f \in C(G), \quad \tau(\lambda_m f) \coloneqq \int_G \delta_{m,0} f(z) \frac{1}{2\pi i} \frac{dz}{z}.$$

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Note that (A_θ, α) is principal since $A_\theta^G = C(G)$ and

$$\forall m \in \mathbb{Z}, \quad (A_\theta)_m = \lambda_m \cdot C(G).$$

Equivariant spectral triples

Let (A, α) be a unital sep'ble G-C^*-algebra. *Let* $n \geqslant \dim G = 1$.

An n-*multigraded* G-*spectral triple* for (A, α) is $(\mathcal{A}, H, D, U)$:

1. (H, U) is a faithful $\mathbb{Z}_2$-graded covariant $*$-representation of $(\mathbb{C}l_n \widehat{\otimes} A, \mathrm{id} \widehat{\otimes} \alpha)$;
2. D is an odd G-invariant self-adjoint operator on H s.t.
$$(D + i)^{-1} \in K(H), \quad [D, \mathbb{C}l_n] = \{0\}, \quad \mathrm{Dom}(D) \subset C^1(H, U);$$
3. $\mathcal{A} \subset A$ is a dense G-invariant $*$-subalgebra s.t.
$$\mathcal{A} \subset C^1(A, \alpha), \quad \mathcal{O}(G) * \mathcal{A} \subseteq \mathcal{A}, \quad [D, \mathcal{A}] \subset B(H).$$

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$$S(A)^2 \ni (\mu, \nu) \mapsto \rho_D(\mu, \nu) \coloneqq \sup_{\substack{a \in \mathcal{A} \\ \|[D,a]\| \leqslant 1}} |\mu(a) - \nu(a)|;$$

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$$(0, +\infty) \ni t \mapsto \exp(-tD^2) \in \mathcal{L}_1(H) \quad \text{(ideally)};$$

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$$(0, +\infty) \ni t \mapsto \exp(-tD^2) \in \mathcal{L}_1(H) \quad \text{(ideally)};$$
4. *index theory* (i.e., NC algebraic topology) via
$$[D] \in KK_n^G(A, \mathbb{C}).$$

Running example

Let $\mathcal{A}_\theta \coloneqq \mathrm{Span}\{\lambda_m \cdot f \mid m \in \mathbb{Z},\ f \in \mathcal{O}(G)\}$.

Since G acts smoothly on $\mathcal{A}_\theta$, get $\partial_1 : \mathcal{A}_\theta \to \mathcal{A}_\theta$ defined by

$$\forall m \in \mathbb{Z}, \forall f \in \mathcal{O}(G), \quad \partial_1(\lambda_m f) \coloneqq d\alpha(\tfrac{\partial}{\partial t})(\lambda_m f) = im \cdot \lambda_m f;$$

since $\mathcal{A}^G = \mathcal{O}(G)$, get $\partial_2 : \mathcal{A}_\theta \to \mathcal{A}_\theta$ defined by

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Let $\gamma_1 \coloneqq \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$ and $\gamma_2 \coloneqq \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, let $\Gamma_{\mathbb{C}^2} \coloneqq \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and let

$$H \coloneqq \mathbb{C}^2 \mathbin{\widehat{\otimes}} \mathbb{C}^2 \otimes L^2(A_\theta, \tau), \quad U \coloneqq \mathrm{id} \mathbin{\widehat{\otimes}} \mathrm{id} \otimes \alpha,$$

$$D \coloneqq \mathrm{id} \mathbin{\widehat{\otimes}} 2\pi \left(\gamma_1 \otimes \partial_1 + \gamma_2 \otimes \partial_2\right).$$

$(\mathcal{A}_\theta, H, D, U)$ is a 2-multigraded G-spectral triple for (A_θ, α).

Definition (cf. Dąbrowski–Sitarz, Forsyth–Rennie)

A *vertical geometry* for $(\mathcal{A}, H, D, U)$ is odd $c(dt) \in B(H)^G$, s.t.

1. $c(dt)^* = -c(dt)$ and $c(dt)^2 = -4\pi^2$,
2. $[c(dt), \mathbb{C}l_n] = [c(dt), \mathcal{A}] = \{0\}$,
3. $\mu(\frac{\partial}{\partial t}) \coloneqq -\frac{1}{2}[D, \frac{1}{4\pi^2}c(dt)] - dU(\frac{\partial}{\partial t}) \in B(H)$.

Vertical geometries and vertical Dirac operators

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We also define $V_1\mathcal{A} \coloneqq \mathbb{C}l_1 \cdot \mathbb{C}[c(dt)] \cdot \mathcal{A}$ and $V_1A \coloneqq \overline{V_1\mathcal{A}}^{B(H)}$.

Remainders and horizontal Dirac operators

Definition

A *remainder* for $(\mathcal{A}, H, D, U; c(dt))$ is $Z \in B(H)^G$ odd, s.t.

$$Z^* = Z, \quad [Z, \mathbb{C}l_n] = 0;$$

its *horizontal Dirac operator* is

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Example

The *canonical remainder* for $(\mathcal{A}, H, D; U; c(dt))$ is

$$Z_{\text{can}} \coloneqq c(dt)\mu(\tfrac{\partial}{\partial t}).$$

Running example

Recall that $D \coloneqq \mathrm{id} \mathbin{\widehat{\otimes}} 2\pi\gamma_1 \otimes \partial_1 + \mathrm{id} \mathbin{\widehat{\otimes}} 2\pi\gamma_2 \otimes \partial_2$.

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Let $c(dt) \coloneqq \mathrm{id} \mathbin{\widehat{\otimes}} 2\pi\gamma_1 \otimes \mathrm{id}$.

Then $c(dt)$ is a vertical geometry for $(\mathcal{A}_\theta, H, D, U)$ with:

- $\mu(\frac{\partial}{\partial t}) = 0$;
- $D_v = \mathrm{id} \mathbin{\widehat{\otimes}} 2\pi\gamma_1 \otimes \partial_1$;
- $V_1\mathcal{A}_\theta = \mathbb{C}[\gamma_1] \mathbin{\widehat{\otimes}} \mathbb{C}[\gamma_1] \widehat{\otimes} \mathcal{A}_\theta$ and $V_1 A_\theta = \mathbb{C}[\gamma_1] \mathbin{\widehat{\otimes}} \mathbb{C}[\gamma_1] \widehat{\otimes} A_\theta$.

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Moreover:

- $Z_{\mathrm{can}} = 0$;
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The vertical geometry $c(dt)$ recovers

$$D = D_v + D_h[Z_{\mathrm{can}}] + Z_{\mathrm{can}} = \mathrm{id} \mathbin{\widehat{\otimes}} 2\pi\gamma_1 \otimes \partial_1 + \mathrm{id} \mathbin{\widehat{\otimes}} 2\pi\gamma_2 \otimes \partial_2 + 0.$$

Principal spectral triples

Definition

If (A, α) is principal, then $(\mathcal{A}, H, D, U; c(dt); Z)$ defines a *principal* G*-spectral triple* for (A, α) whenever

1. $\overline{V_1 A \cdot H^G} = H$;
2. $[D_h[Z], \mathcal{A}] \subset \overline{A \cdot [D - Z, \mathcal{A}^G]}^{B(H)}$;
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Examples (cf. Brain–Mesland–Van Suijlekom)

1. $(\mathcal{A}_\theta, H, D, U; c(dt); 0)$ for $(\mathcal{A}_\theta, \alpha)$, where
$$c(dt) := \mathrm{id} \,\widehat{\otimes}\, 2\pi\gamma_1 \otimes \mathrm{id}, \quad 0 = Z_{\mathrm{can}};$$
2. the canonical spectral triple for $C(S^3_\theta)$.

Analysis

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1. $c(dt)$ encodes orbitwise intrinsic geometry, index theory via the *wrong-way* cycle (cf. Wahl)

$$(\mathcal{A}, E_1, S_1, W_1) \coloneqq (\mathcal{A}, L^2(V_1 A, \mathbb{E}_{V_1 A^G}), c(dt)\, d\alpha(\tfrac{\partial}{\partial t}), \alpha) \in \Psi_1^G;$$

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2. $D^G[Z] \coloneqq D_{\mathfrak{h}}[Z]|_{H^G}$ encodes basic geometry, index theory via the *basic* spectral triple $(V_1\mathcal{A}^G, H^G, D^G[Z]) \in \Psi_{n-1}^G$;

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Note (cf. Carey–Neshveyev–Nest–Rennie, Arici–Kaad–Landi...)

Since $G = U(1)$, the cycle $(\mathcal{A}, E_1, S_1)$ represents the extension class $[\partial] \in KK_1(A, A^G)$ of A as a Pimsner algebra.

Synthesis

Theorem

Let $(\mathcal{A}, H, D, U; c(dt); Z)$ *be a principal* G*-spectral triple:*

1. $H \cong E_1 \widehat{\otimes}_{V_1 A^G} H^G$ *and* $D_v = S_1 \widehat{\otimes}\, \mathrm{id}$;
2. $[D_h[Z], \cdot]$ *canonically induces a Hermitian connection* $\nabla[Z]$ *on* E_1 *s.t.* $D_h[Z] = \mathrm{id}\, \widehat{\otimes}_{\nabla[Z]} D^G[Z]$;
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3. $[D] = [S_1] \otimes_{V_1 A^G} [D^G[Z]]$ *in* G*-equivariant* KK*-theory.*

Thus, in G-equivariant unbounded KK-theory,

$$(\mathcal{A}, H, D - Z, U) \cong (\mathcal{A}, E_1, S_1, W_1; \nabla[Z]) \widehat{\otimes}_{V_1 \mathcal{A}^G} (V_1 \mathcal{A}^G, H^G, D^G[Z], \mathrm{id}).$$

Running example

Recall that $A_\theta \coloneqq C(G) \rtimes_\theta \mathbb{Z}$; note that $V_1 A_\theta^G \cong M_2(\mathbb{C}) \otimes C(G)$.

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1. The wrong-way cycle

   $$(\mathcal{A}_\theta, \mathbb{C}[\gamma_1] \widehat{\otimes} \mathbb{C}[\gamma_1] \otimes L^2(A_\theta, \mathbb{E}_{C(G)}), \mathrm{id} \widehat{\otimes} 2\pi\gamma_1 \otimes \partial_1, \mathrm{id} \widehat{\otimes} \mathrm{id} \otimes \alpha)$$

   represents the connecting map $K_i(A_\theta) \to K_{i+1}(C(G))$.

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Running example

Recall that $A_\theta \coloneqq C(G) \rtimes_\theta \mathbb{Z}$; note that $V_1 A_\theta^G \cong M_2(\mathbb{C}) \otimes C(G)$.

1. The wrong-way cycle

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3. $[D_h[0], c(dt)] = 0$ encodes totally geodesic orbits.
4. $[D_h[0], \cdot]$ on $\mathcal{A}_\theta$ gives a horizontal lift of the de Rham calculus on $\mathcal{O}(G) = \mathcal{A}_\theta^G$.

But wait, there's more!

Get the space $\mathfrak{At}$ of NC principal connections by varying $\mathrm{D}_{\mathfrak{h}}[Z]$ while fixing:

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Theorem

1. $\mathfrak{At}$ *is an affine space;*
2. $\mathfrak{G}$ *acts on* $\mathfrak{At}$ *by affine transformations;*
3. $[D_{v} + D_{h}[Z]] \in KK^{G}_{n}(A, \mathbb{C})$ *is constant in* $D_{h}[Z] \in \mathfrak{At}$.

This generalises the commutative case (up to a cocycle).

Running example

$A_\theta \coloneqq C(G) \rtimes_\theta \mathbb{Z} \hookleftarrow C(G) = A_\theta^G$ is a trivial NC principal bundle.

$(\mathcal{A}_\theta, H, D, U; c(dt); 0)$ admits non-trivial NC gauge theory:

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with $\mathfrak{s} \triangleright (\text{basepoint} + \mathfrak{a}) = \text{basepoint} + (\mathfrak{a} + \mathfrak{s} d \mathfrak{s}^*)$.

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This NC gauge theory is highly sensitive to the value of θ.