REVISITING THE DIFFERENTIAL TOPOLOGY OF NONCOMMUTATIVE TORI

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Shameless self-promotion

1. B. Ć., *Geometric foundations for classical* U(1)*-gauge theory on noncommutative manifolds*, Commun. Math. Phys. 405 (2024), article no. 209.
2. B. Ć. and Timmavajjula Venkata Karthik, *Maxwell's equations on noncommutative oriented Riemannian manifolds*, to appear.
3. B. Ć., *Classification of* U(1)*-instantons on higher-dimensional noncommutative tori*, in preparation.

1. Review of noncommutative tori

Basic definitions

Fix $N \geq 2$ and $\theta \in \Lambda^2 \mathbf{R}^N$.

The C*-algebraic *noncommutative N-torus* $\mathfrak{A}_\theta$ is the universal C*-algebra with unitary generators $\{U_\mathbf{m} : \mathbf{m} \in \mathbf{Z}^{*N}\}$ and relations

$$U_\mathbf{m} \star_\theta U_\mathbf{n} = \exp(-\pi i \langle \mathbf{m} \wedge \mathbf{n}, \theta \rangle) U_{\mathbf{m}+\mathbf{n}}.$$

If $N = 2$, then $\mathfrak{A}_\theta \simeq C(\mathbf{S}^1) \rtimes_R \mathbf{Z}$, where R is rotation by $2\pi\langle e^1 \wedge e^2, \theta \rangle$ radians.

The canonical $\mathbf{T}^N$-action α and $\mathbf{T}^N$-invariant faithful trace τ on $\mathfrak{A}_\theta$ are defined by

$$\alpha_t(U_\mathbf{m}) := \exp(2\pi i \langle \mathbf{m}, t \rangle) U_\mathbf{m}, \qquad \tau(U_\mathbf{m}) := \delta_{0,\mathbf{m}}.$$

The C*-algebra $\mathfrak{A}_\theta$ is simple iff the alternating bicharacter $\sigma_\theta : \mathbf{Z}^{*N} \times \mathbf{Z}^{*N} \to \mathbf{T}$ given by

$$\sigma_\theta(\mathbf{m}, \mathbf{n}) := \langle \mathbf{m} \wedge \mathbf{n}, \theta \rangle \bmod \mathbf{Z}$$

is nondegenerate, in which case we call θ *completely irrational*.

If $N = 2$, then θ is completely irrational iff $\langle e^1 \wedge e^2, \theta \rangle \notin \mathbf{Q}$.

Theorem (Blackadar-Kumjian-Rørdam; Elliott-Evans, Q. Lin, Boca, Kishimoto, Phillips)
If θ is completely irrational, then $\mathfrak{A}_\theta$ is a simple unital A$\mathbf{T}$-algebra of real rank 0 and stable rank 1, whence it is classifiable by its scaled ordered K-theory

$$(K_0(\mathfrak{A}_\theta), K_0(\mathfrak{A}_\theta)^+, [\mathfrak{A}_\theta], K_1(\mathfrak{A}_\theta)).$$

The *smooth noncommutative N-torus* is the local C*-algebra

$$A_\theta := \{a \in \mathfrak{A}_\theta : (t \mapsto \alpha_t(a)) \in C^\infty(\mathbf{T}^N, \mathfrak{A}_\theta)\}.$$

The canonical $*$-differential calculus on A_θ is $(\Omega_\theta, \mathrm{d})$, where

$$\Omega_\theta := A_\theta \otimes \Lambda \mathbf{C}^{*N}, \qquad \mathrm{d}(U_{\mathbf{m}} \otimes \omega) := U_{\mathbf{m}} \otimes 2\pi\mathrm{i}\mathbf{m} \wedge \omega.$$

Theorem (Connes, cf. Pimsner)

We have an explicit Chern character $\mathrm{Ch} : K_\bullet(A_\theta) \to \Lambda^{\mathrm{ev/odd}} \mathbf{C}^{*N}$ *given by*

$$\mathrm{Ch}([p]) = \sum_{k \geq 0} \frac{1}{k!} \frac{1}{(2\pi\mathrm{i})^k} (\tau \otimes \mathrm{id})\left(p \star_\theta (\mathrm{d}p)^{\star_\theta (2k)}\right),$$

$$\mathrm{Ch}([u]) = \sum_{k \geq 0} \frac{k!}{(2k+1)!} \frac{1}{(2\pi\mathrm{i})^{k+1}} (\tau \otimes \mathrm{id})\left((u^* \star_\theta \mathrm{d}u) \star_\theta (\mathrm{d}u^* \star_\theta \mathrm{d}u)^{\star_\theta k}\right).$$

Theorem (Connes, cf. Bost)

The inclusion map $A_\theta \hookrightarrow \mathfrak{A}_\theta$ induces a strong isomorphism in K-theory.

Theorem (Elliott)

There is a unique isomorphism $\mu : K_\bullet(\mathfrak{A}_\theta) \to \Lambda^{\mathrm{ev}/\mathrm{odd}}\mathbf{Z}^{*N}$ such that

$$\langle \mu(c), X_1 \wedge \cdots \wedge X_p \rangle = \langle \mathrm{Ch}(c), \mathrm{e}^{-\theta} \wedge X_1 \wedge \cdots \wedge X_p \rangle,$$

whence $\mu([\mathfrak{A}_\theta]) = 1$ and $\mathrm{rk}_T = \langle \mu(\cdot), \mathrm{e}^{\theta} \rangle$.

Theorem (Rieffel)

If θ is completely irrational, then $\mathfrak{A}_\theta$ has cancellation and

$$\mu(K_0(\mathfrak{A}_\theta)^+) = \{\beta \in \Lambda\mathbf{Z}^{*N} : \langle \beta, \mathrm{e}^{\theta} \rangle > 0\} \cup \{0\}.$$

2. Classification of symmetries

Homeomorphisms

For all $\theta_1, \theta_2 \in \Lambda^2 \mathbf{R}^N$, set

$$\mathfrak{G}_1(\theta_1, \theta_2) := \{\gamma \in \mathrm{GL}^{\mathrm{ev}}(\Lambda \mathbf{Z}^N) : \gamma(1) = 1,\ \gamma(e^{\theta_1}) = e^{\theta_2}\}.$$

If $N = 2$ and $\langle e^1 \wedge e^2, \theta \rangle \notin \mathbf{Q}$, then $\mathfrak{G}_1(\theta, \theta) \cong \mathrm{GL}(2, \mathbf{Z})$ via $\gamma \mapsto \gamma\restriction_{\mathbf{Z}^2}$.

Theorem (Phillips)

If θ_1 and θ_2 are completely irrational, then $\mathfrak{A}_{\theta_1} \simeq \mathfrak{A}_{\theta_2}$ iff $\mathfrak{G}_1(\theta_1, \theta_2)$ is nonempty.

Theorem (Elliott, cf. Elliott–Rørdam)

If θ is completely irrational, then we have the group extension

$$1 \to \overline{\mathrm{Inn}}(\mathfrak{A}_\theta) \to \mathrm{Aut}(\mathfrak{A}_\theta) \xrightarrow{M} \mathfrak{G}_1(\theta, \theta) \to 1,$$

where $M(\varphi)^{-\mathsf{T}} = \mu \circ K_\bullet(\varphi) \circ \mu^{-1}$.

Given $\theta_1, \theta_2 \in \Lambda^2 \mathbf{R}^N$, set

$$G_1(\theta_1, \theta_2) := \{g \in \mathrm{GL}(N, \mathbf{Z}) : \theta_2 - \Lambda^2 g(\theta_1) \in \Lambda^2 \mathbf{Z}^N\}.$$

If $N = 2$ and $\langle e^1 \wedge e^2, \theta \rangle \notin \mathbf{Q}$, then $G_1(\theta, \theta) = \mathrm{SL}(2, \mathbf{Z}) < \mathrm{GL}(2, \mathbf{Z}) \simeq \mathfrak{G}_1(\theta, \theta)$.

Theorem (Cuntz–Elliott–Goodman–Jørgensen)

If θ_1 and θ_2 are completely irrational and Diophantine, then $A_{\theta_1} \simeq A_{\theta_2}$ iff $G_1(\theta_1, \theta_2)$ is nonempty.

Theorem (Elliott)

If $N = 2$ and $\langle e^1 \wedge e^2, \theta \rangle$ is Diophantine-irrational, then

$$\mathrm{Aut}(A_\theta) \simeq (\mathrm{PU}(A_\theta)_1 \rtimes \mathbf{T}^2) \rtimes \mathrm{SL}(2; \mathbf{Z}),$$

$$\mathrm{Out}(A_\theta) \simeq \frac{\mathbf{T}^2}{\exp_{\mathbf{T}^2}(\langle e^1 \wedge e^2, \theta \rangle \cdot \mathbf{Z}^2)} \rtimes \mathrm{SL}(2; \mathbf{Z}).$$

What is a noncommutative diffeomorphism?

We take a *noncommutative manifold* to be $(B; \Omega_B, \mathrm{d})$, where:

1. B is a unital pre-C*-algebra; and
2. (Ω, d) is a $*$-differential calculus on B.

Considerations of U(1) gauge theory in [Ć. 2024] motivate

$$\mathrm{DAut}(B) := \{(\beta, \varphi) \in \Omega^1_{\mathrm{sa}} \rtimes \mathrm{Aut}(\Omega) : \mathrm{d} - \varphi \circ \mathrm{d} \circ \varphi^{-1} = \mathrm{i}[\beta, \cdot]\},$$
$$\mathrm{DOut}(B) := \mathrm{DAut}(B)/\{(-\mathrm{i}\,\mathrm{d}u \cdot u^*, \mathrm{Ad}_u) : u \in \mathrm{U}(B)\}.$$

We view $\mathrm{DAut}(B)$ as the automorphism group of $(B; \Omega_B, \mathrm{d}_B)$ in a suitable category of noncommutative manifolds and 'extended diffeomorphisms.'

Diffeomorphisms redux

Theorem (Cuntz-Elliott-Goodman-Jørgensen $+\,\varepsilon$)

If θ_1 and θ_2 are completely irrational, then $(A_{\theta_1}; \Omega_{\theta_1}, \mathrm{d}) \simeq (A_{\theta_2}; \Omega_{\theta_2}, \mathrm{d})$ iff $G_1(\theta_1, \theta_2)$ is nonempty.

Theorem (Ć.–Venkata Karthik)

If θ is nondegenerate, then we have group extensions

$$1 \to \mathbf{R}^{*N} \times (\mathrm{PU}(A_\theta)^0 \rtimes \mathbf{T}^N) \to \mathrm{DAut}(A_\theta) \xrightarrow{D} G_1(\theta, \theta) \to 1,$$

$$1 \to \frac{\mathbf{R}^{*N} \oplus \mathbf{R}^N}{\exp_{\mathrm{SO}(N,N)}(\theta)(\mathbf{Z}^{*N} \oplus \mathbf{Z}^N)} \to \mathrm{DOut}(A_\theta) \xrightarrow{D} G_1(\theta, \theta) \to 1,$$

where $\mathrm{PU}(A_\theta)^0 \coloneqq \{[u] \in \mathrm{PU}(A_\theta) : \mu([u])_{(1)} = 0\}$.

3. Classification of line bundles

Line bundles over noncommutative spaces

Strong Morita self-equivalences of a unital C*-algebra $\mathfrak{B}$ yield a group $\mathrm{Pic}(\mathfrak{B})$ satisfying

$$\{[\mathfrak{E}] \in \mathrm{Pic}(\mathfrak{B}) : \mathfrak{E} \simeq \mathfrak{B} \text{ as right Hilbert } \mathfrak{B}\text{-modules}\} \simeq \mathrm{Out}(\mathfrak{B}).$$

In [Ć. 2024], Hermitian line bimodules with Hermitian bimodule connection over a noncommutative manifold $(B; \Omega, \mathrm{d})$ yield a group $\mathrm{DPic}(B)$ satisfying

$$\{[E, \nabla] \in \mathrm{DPic}(B) : E \simeq B \text{ as right pre-Hilbert } B\text{-modules}\} \simeq \mathrm{DOut}(B).$$

If θ is completely irrational, then $\mathrm{DPic}(A_\theta)$ is the automorphism group of $(A_\theta; \Omega_\theta, \mathrm{d})$ in the category of *complete Morita equivalences* à la Schwarz (up to unitary equivalence).

Topological line bundles

For all $\theta_1, \theta_2 \in \Lambda^2 \mathbf{R}^N$, set

$$\mathfrak{G}(\theta_1, \theta_2) \coloneqq \{g \in \mathrm{GL}^{\mathrm{ev}}(\Lambda \mathbf{Z}^N) : \gamma(e^{\theta_1}) \in \mathbf{R}_{>0} \cdot e^{\theta_2}\}.$$

If $N = 2$, then $\mathfrak{G}(\theta_1, \theta_2) \neq \emptyset$ iff $\langle e^1 \wedge e^2, \theta_2 \rangle = \frac{a\langle e^1 \wedge e^2, \theta_1 \rangle + b}{c\langle e^1 \wedge e^2, \theta_1 \rangle + d}$ for some $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}(2, \mathbf{Z})$.

Theorem (Phillips)

If θ_1 and θ_2 are completely irrational, then $\mathfrak{A}_{\theta_1}$ and $\mathfrak{A}_{\theta_2}$ are strongly Morita equivalent iff $\mathfrak{G}(\theta_1, \theta_2)$ is nonempty.

Theorem (Phillips $+ \varepsilon$)

If θ is completely irrational, then we have the group extension

$$1 \to \overline{\mathrm{Inn}}(\mathfrak{A}_\theta) / \mathrm{Inn}(\mathfrak{A}_\theta) \to \mathrm{Pic}(\mathfrak{A}_\theta) \xrightarrow{M} \mathfrak{G}(\theta, \theta) \to 1,$$

where $M([E])^{\mathsf{T}} \mu(\mathbf{c}) = \mu(\mathbf{c} \otimes_{\mathfrak{A}_\theta} [E])$.

Line bundles with connection

For all $\theta_1, \theta_2 \in \Lambda^2 \mathbf{R}^N$, set

$$G(\theta_1, \theta_2) := \{x \in \mathrm{Spin}(N,N) : \rho(x) \Lambda \mathbf{Z}^N = \Lambda \mathbf{Z}^N,\ \rho(x) e^{\theta_1} \in \mathbf{R}_{>0} \cdot e^{\theta_2}\}.$$

If $N = 2$, then $G(\theta_1, \theta_2) \neq \emptyset$ iff $\mathfrak{G}(\theta_1, \theta_2) \neq \emptyset$.

Theorem (Schwarz, Rieffel–Schwarz, Li)

Noncommutative N-tori A_{θ_1} and A_{θ_2} are completely Morita equivalent iff $G(\theta_1, \theta_2)$ is nonempty.

Theorem (Ć.)

If θ is completely irrational, then we have the group extension

$$1 \to \frac{\mathbf{R}^{*N} \oplus \mathbf{R}^N}{\exp_{SO(N,N)}(\theta)(\mathbf{Z}^{*N} \oplus \mathbf{Z}^N)} \to \mathrm{DPic}(A_\theta) \to G(\theta, \theta) \to 1.$$

Topological line bundles via K-theoretic rank

The rank function $\mathrm{rk}_\tau : K_0(\mathfrak{A}_\theta) \to \mathbf{R}$ is injective iff $\langle \cdot, e^\theta \rangle : \Lambda^{\mathrm{ev}} \mathbf{Q}^{*N} \to \mathbf{R}$ is injective, in which case we call θ *totally irrational*.

If θ is totally irrational, given a subring R of $\mathbf{Q}$, let

$$\mathcal{A}_\theta[R] := \{ r \in \langle \Lambda^{\mathrm{ev}} R^{*N}, e^\theta \rangle : r \cdot \langle \Lambda^{\mathrm{ev}} R^{*N}, e^\theta \rangle \subseteq \langle \Lambda^{\mathrm{ev}} R^{*N}, e^\theta \rangle \}.$$

Proposition (Iglesias–Lachaud)

If θ is totally irrational, then $\mathbf{K}_\theta := \mathcal{A}_\theta[\mathbf{Q}]$ is a real number field satisfying $[\mathbf{K}_\theta : \mathbf{Q}] \mid 2^{N-1}$ and $\mathcal{O}_\theta := \mathcal{A}_\theta[\mathbf{Z}]$ is an order in $\mathbf{K}_\theta$.

Theorem (Nawata–Watatani)

If θ is totally irrational, then we have the group extension

$$1 \to \mathrm{Out}(\mathfrak{A}_\theta) \to \mathrm{Pic}(\mathfrak{A}_\theta) \xrightarrow{\mathrm{rk}_\tau} \mathcal{O}_\theta^\times \cap \mathbf{R}_{>0} \to 1.$$

Differentiable vs topological line bundles when $N \geq 3$

Theorem (Ć.)

If $N \geq 3$, θ is totally irrational, and $[\mathbf{K}_\theta : \mathbf{Q}] = 2^{N-1}$, then

$$1 \to \mathrm{DOut}(A_\theta) \to \mathrm{DPic}(A_\theta) \xrightarrow{\mathrm{rk}_\tau} \{r \in \mathcal{O}_\theta^\times \cap \mathbf{R}_{>0} : \mathcal{N}_{\mathbf{K}_\theta/\mathbf{Q}}(r) = 1\}$$

is an exact sequence that is short-exact when $N = 3$.

Example

Set $N = 3$ and choose totally irrational θ such that $\mathbf{K}_\theta = \mathbf{Q}(\sqrt{5}, \sqrt{13})$ and $\mathcal{O}_\theta$ is its ring of integers, e.g.,

$$\theta := \tfrac{1+\sqrt{5}}{2} e_1 \wedge e_2 + \tfrac{1+\sqrt{13}}{2} e_1 \wedge e_3 + \tfrac{1+\sqrt{5}+\sqrt{13}+\sqrt{65}}{4} e_2 \wedge e_3.$$

The unit $\frac{9+\sqrt{5}+\sqrt{13}+\sqrt{65}}{4} \in \mathcal{O}_\theta^\times \cap \mathbf{R}_{>0}$ of norm -1 lifts to $\mathrm{Pic}(\mathfrak{A}_\theta)$ but not to $\mathrm{DPic}(A_\theta)$.

Suppose that $N = 2$ and $\langle e^1 \wedge e^2, \theta\rangle \notin \mathbf{Q}$, whence θ is totally irrational.

1. If $\langle e^1 \wedge e^2, \theta\rangle$ is quadratic, then $\mathbf{K}_\theta = \mathbf{Q}(\langle e^1 \wedge e^2, \theta\rangle)$ and $\mathcal{O}_\theta^\times \cap \mathbf{R}_{>0} \simeq \mathbf{Z}$.
2. Else, $\mathbf{K}_\theta = \mathbf{Q}$ and $\mathcal{O}_\theta = \mathbf{Z}$, whence $\mathcal{O}_\theta^\times \cap \mathbf{R}_{>0} = 1$.

Theorem (Ć.)

1. *We have the group extension*

$$1 \to \mathrm{DOut}(A_\theta) \to \mathrm{DPic}(A_\theta) \xrightarrow{\mathrm{rk}_\tau} \mathcal{O}_\theta^\times \cap \mathbf{R}_{>0} \to 1.$$

2. *If $\langle e^1 \wedge e^2, \theta\rangle$ is quadratic and $r \in \mathcal{O}_\theta^\times \cap \mathbf{R}_{>0} \setminus \{1\}$, then the 'real multiplication instanton' corresponding to the 'basic Heisenberg module' $[E, \nabla] \in \mathrm{DPic}(A_\theta)$ of rank r behaves like the q-monopole for $q = r$.*

Behaving like the q-monopole

1. The datum (E, ∇) corresponds to an essentially unique *horizontally differentiable quantum principal* $\mathrm{U}(1)$*-bundle* $(P; \Omega_{P,\mathrm{hor}}, \mathrm{d}_{P,\mathrm{hor}})$ over $(A_\theta; \Omega_\theta, \mathrm{d})$.
2. Since (E, ∇) has *vertical deformation parameter* q^2, $(P; \Omega_{P,\mathrm{hor}}, \mathrm{d}_{P,\mathrm{hor}})$ only completes to an essentially unique q^2*-differentiable quantum principal* $\mathrm{U}(1)$*-bundle with connection* $(P; \Omega_P, \mathrm{d}_P; \Pi)$ over $(A_\theta; \Omega_\theta, \mathrm{d})$.
3. The canonical spectral triple for $(A_\theta; \Omega_\theta, \mathrm{d})$ lifts to an essentially unique *faithful projectable commutator representation* of $(P; \Omega_P, \mathrm{d}_P; \Pi)$ with unique *vertical twist* (yielding powers of q^2) and *horizontal twist* (yielding powers of q).
4. This canonically induces a $\mathrm{U}(1)$-invariant Lipschitz seminorm à la Kaad–Kyed for $P := A_\theta \rtimes_E^{\mathrm{alg}} \mathbf{Z}$ with $t = q^2$.

We shall not cease from exploration
And the end of all our exploring
Will be to arrive where we started
And know the place for the first time.

—T. S. Eliot, "Little Gidding"