Principal bundles in NC Riemannian geometry

Branimir Ćaćić [1] and Bram Mesland [2]

Quantum Flag Manifolds in Prague, September 2019

[1]Department of Mathematics & Statistics, University of New Brunswick

[2]Mathematical Institute, Leiden University

Coming soon to an arXiv near you...

B. Ć. and B. Mesland, *Gauge theory on noncommutative Riemannian principal bundles*

Why bother?

What is a NC gauge theory with compact connected Lie structure group K?

Why bother?

What is a NC gauge theory with compact connected Lie structure group K?

Thesis (Brzeziński–Majid, Hajac, et al.)

Strong connections on principal $\mathcal{O}(K)$-comodule algebras.

Why bother?

What is a NC gauge theory with compact connected Lie structure group K?

Thesis (Brzeziński–Majid, Hajac, et al.)

Strong connections on principal $\mathcal{O}(K)$-comodule algebras.

Antithesis (Chamseddine–Connes et al.)

The spectral action principle applied to suitable spectral triples.

Why bother?

What is a NC gauge theory with compact connected Lie structure group K?

Thesis (Brzeziński–Majid, Hajac, et al.)

Strong connections on principal $\mathcal{O}(K)$-comodule algebras.

Antithesis (Chamseddine–Connes et al.)

The spectral action principle applied to suitable spectral triples.

Synthesis?

Unbounded KK-theory in the spirit of Brain–Mesland–Van Suijlekom, but cf. Conv{Dąbrowski, Sitarz, Zucca}.

Spectral triples

The quantum Weyl algebra

Let $\mathfrak{g}$ be a positive quadratic Lie algebra over $\mathbb{R}$.

Definition (Alekseev–Meinrenken, cf. Kostant)

The *quantum Weyl algebra* is the unital $*$-algebra $\mathcal{W}(\mathfrak{g})$ over $\mathbb{R}$ with even skew-adjoint generators in $\mathfrak{g}$ and odd skew-adjoint generators in $\mathfrak{g}^*$, satisfying:

1. $\forall \alpha, \beta \in \mathfrak{g}^*,\ [\alpha, \beta] = -2\langle \alpha, \beta \rangle$;
2. $\forall X, Y \in \mathfrak{g},\ [X, Y] = [X, Y]_{\mathfrak{g}} = \mathrm{ad}(X)Y$;
3. $\forall X \in \mathfrak{g}, \forall \alpha \in \mathfrak{g}^*,\ [X, \alpha] = \mathrm{ad}^*(X)\alpha$.

In other words, $\mathcal{W}(\mathfrak{g}) = \mathrm{Cl}(\mathfrak{g}^*) \rtimes_{\mathrm{ad}^*} \mathcal{U}(\mathfrak{g})$.

The cubic Dirac element

Definition (Kostant)

The *cubic Dirac element* is the odd self-adjoint element

$$\mathcal{D}_{\mathfrak{g}} := \varepsilon^i \varepsilon_i + \frac{1}{6}\langle \varepsilon_i, [\varepsilon_j, \varepsilon_k]\rangle \varepsilon^i \varepsilon^j \varepsilon^k \in \mathcal{W}(\mathfrak{g}). \tag{1}$$

It turns out that $\mathcal{D}_{\mathfrak{g}}$ satisfies the following:

1. $\forall \alpha \in \mathfrak{g}^*$, $[\mathcal{D}_{\mathfrak{g}}, \alpha] = -2\alpha^\sharp$, where $\alpha^\sharp := \langle \alpha, \varepsilon^i \rangle \varepsilon_i$;
2. $\forall X \in \mathfrak{g}$, $[\mathcal{D}_{\mathfrak{g}}, X] = 0$;
3. $\mathcal{D}_{\mathfrak{g}}^2 \equiv \Delta_{\mathfrak{g}} \bmod \mathrm{Cl}(\mathfrak{g}^*) + \mathrm{Cl}(\mathfrak{g}^*) \cdot \mathfrak{g}$, where $\Delta_{\mathfrak{g}} := -\langle \varepsilon^i, \varepsilon^j \rangle \varepsilon_i \varepsilon_j$.

Note, in particular, that $\mathcal{D}_{\mathfrak{g}}^2$ is central.

Differential operators

Suppose that $\mathfrak{g}$ integrates to a connected Lie group G.

Let $c^V : \mathrm{Cl}(\mathfrak{g}^*) \to B(V)$ be a G-equivariant $\mathbf{Z}/2$-graded finite-dimensional $*$-representation.

Let $U^V : G \to U(L^2(G,V))$ be the resulting unitary representation.

Note that the G-equivariant identification of $\mathfrak{g}$ with left-fundamental (i.e., right-invariant) vector fields on G yields

$$\mathfrak{g} \times G \xrightarrow{\sim} TG, \quad \mathfrak{g}^* \times G \xrightarrow{\sim} T^*G, \quad \mathrm{Cl}(\mathfrak{g}^*) \times G \xrightarrow{\sim} \mathrm{Cl}(T^*G).$$

Thus, c^V extends (via dU^V on $\mathfrak{g}$) to a G-equivariant $\mathbf{Z}/2$-graded $*$-representation of $\mathcal{W}(\mathfrak{g})$ by differential operators on $L^2(G,V)$.

Spectral triples

Suppose—purely for simplicity—that G is compact.

Let $(\mathcal{A}, H, D) := (C^\infty(G), L^2(G, V), c^V(\mathcal{D}_\mathfrak{g}))$. Then:

1. H is a $\mathbf{Z}/2$-graded separable Hilbert space;
2. D is a densely-defined odd self-adjoint operator on H with
$$(D + i)^{-1} \in K(H);$$
3. $\mathcal{A}$ is a unital $*$-subalgebra of $B(H)$, such that
$$\forall a \in \mathcal{A}, \quad a \operatorname{Dom} D \subset \operatorname{Dom} D, \quad [D, a] \in B(H).$$

In other words, $(\mathcal{A}, H, D)$ is a *spectral triple*.

Spectral triples

Suppose—purely for simplicity—that G is compact.

Let $(\mathcal{A}, H, D) := (C^\infty(G), L^2(G, V), c^V(\mathcal{D}_{\mathfrak{g}}))$. Then:

1. H is a $\mathbf{Z}/2$-graded separable Hilbert space;
2. D is a densely-defined odd self-adjoint operator on H with
$$(D + i)^{-1} \in K(H);$$
3. $\mathcal{A}$ is a unital $*$-subalgebra of $B(H)$, such that
$$\forall a \in \mathcal{A}, \quad a \operatorname{Dom} D \subset \operatorname{Dom} D, \quad [D, a] \in B(H).$$

In other words, $(\mathcal{A}, H, D)$ is a *spectral triple*.

Remarks

1. Everything we do can be done in the non-unital case.
2. We want $(\mathcal{A}, H, D)$ to be n-multigraded, i.e., for $n := \dim \mathfrak{g}$.

What are they good for?

The spectral triple $(\mathcal{A}, H, D)$ encodes the following:

1. *first-order (de Rham) differential calculus* via

$$\mathcal{A} \ni a \mapsto [D, a] = c(da);$$

2. *spectral geometry* (e.g., dimension, volume, measure) via

$$(0, +\infty) \ni t \mapsto \exp(-tD^2) \in \mathcal{L}_1(H);$$

3. *index theory* (i.e., NC algebraic topology) via

$$[D] \in K^n(A), \quad A := \overline{\mathcal{A}}^{L(H)} = C(G).$$

What are they good for?

The spectral triple $(\mathcal{A}, H, D)$ encodes the following:

1. *first-order (de Rham) differential calculus* via

$$\mathcal{A} \ni a \mapsto [D, a] = c(da);$$

2. *spectral geometry* (e.g., dimension, volume, measure) via

$$(0, +\infty) \ni t \mapsto \exp(-tD^2) \in \mathcal{L}_1(H);$$

3. *index theory* (i.e., NC algebraic topology) via

$$[D] \in K^n(A), \quad A := \overline{\mathcal{A}}^{L(H)} = C(G).$$

Points 1 and 2 hint at possibilities for NC *gauge theory*.

Principal K-spectral triples

Let $\mathfrak{k} \subset \mathfrak{g}$ be a Lie sub-algebra, so that $\mathcal{W}(\mathfrak{k})$ can identified with the unital $*$-subalgebra of $\mathcal{W}(\mathfrak{g})$ generated by $\mathfrak{k}$ and $\mathfrak{k}^* \cong (\mathfrak{k}^0)^\perp$.

Definition (Kostant)

The *relative cubic Dirac element* of $(\mathfrak{g}, \mathfrak{k})$ is the element

$$\mathcal{D}_{\mathfrak{g},\mathfrak{k}} := \mathcal{D}_{\mathfrak{g}} - \mathcal{D}_{\mathfrak{k}}. \tag{2}$$

It turns out that $\mathcal{D}_{\mathfrak{g},\mathfrak{k}}$ satisfies the following:

1. $\forall \alpha \in \mathfrak{k}^*$, $[\mathcal{D}_{\mathfrak{g},\mathfrak{k}}, \alpha] = 0$;
2. $\forall X \in \mathfrak{k}$, $[\mathcal{D}_{\mathfrak{g},\mathfrak{k}}, X] = 0$.

It follows that $[\mathcal{D}_{\mathfrak{k}}, \mathcal{D}_{\mathfrak{g},\mathfrak{k}}] = 0$ and hence $\mathcal{D}_{\mathfrak{g}}^2 = \mathcal{D}_{\mathfrak{k}}^2 + \mathcal{D}_{\mathfrak{g},\mathfrak{k}}^2$.

Homogeneous spaces

Suppose now that $\mathfrak{k}$ integrates to a compact connected subgroup K of G, so that $\pi : G \to K\backslash G$ is a principal K-bundle.

Observe that $\mathfrak{m} := \mathfrak{k}^{\perp}$ satisfies $[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}$, so that $K\backslash G$ is a reductive homogeneous space.

Since $\mathcal{D}_{\mathfrak{g},\mathfrak{k}}$ is K-invariant, it follows that $c^V(\mathcal{D}_{\mathfrak{g},\mathfrak{k}})$ descends to a differential operator D^K on $L^2(K\backslash G, V \times_K G) \cong L^2(G, V)^K = H^K$.

In fact, it turns out that D^K, like D, is a *Dirac-type operator*, so that $((\mathrm{Cl}(\mathfrak{k}^*) \otimes \mathcal{A})^K, H^K, D^K)$ is a spectral triple.

Homogeneous spaces

Suppose now that $\mathfrak{k}$ integrates to a compact connected subgroup K of G, so that $\pi : G \to K\backslash G$ is a principal K-bundle.

Observe that $\mathfrak{m} := \mathfrak{k}^\perp$ satisfies $[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}$, so that $K\backslash G$ is a reductive homogeneous space.

Since $\mathcal{D}_{\mathfrak{g},\mathfrak{k}}$ is K-invariant, it follows that $c^V(\mathcal{D}_{\mathfrak{g},\mathfrak{k}})$ descends to a differential operator D^K on $L^2(K\backslash G, V \times_K G) \cong L^2(G, V)^K = H^K$.

In fact, it turns out that D^K, like D, is a *Dirac-type operator*, so that $((\mathrm{Cl}(\mathfrak{k}^*) \otimes \mathcal{A})^K, H^K, D^K)$ is a spectral triple.

Question

How are $(\mathcal{A}, H, D)$ and $((\mathrm{Cl}(\mathfrak{k}^*) \otimes \mathcal{A})^K, H^K, D^K)$ related?

Equivariant spectral triples

Let K be a compact connected Lie group.

A *K-spectral triple* consists of a spectral triple $(\mathcal{A}, H, D)$ and an even unitary representation $U : K \to U(H)$, such that:

1. $\mathcal{A}$ is a K-invariant subalgebra of C^1-vectors for
$$\alpha : K \to \mathrm{Aut}(B(H)), \quad k \mapsto (T \mapsto U_k T U_k^*);$$
2. $\mathrm{Dom}\, D$ is a K-invariant subspace of C^1-vectors for U;
3. D is K-invariant.

Running Example

We have $(\mathcal{A}, H, D; U) := (C^\infty(G), L^2(G, V), c^V(\mathcal{D}_\mathfrak{g}); U^V|_K)$.

Vertical geometries

Fix a normalised Ad-invariant inner product $\langle \cdot , \cdot \rangle$ for $\mathfrak{t}$.

A *vertical Clifford action* is a G-equivariant $\mathbf{Z}/2$-graded $*$-representation $c : \mathrm{Cl}(\mathfrak{t}^*) \to B(H)$, such that:

1. $\forall x \in \mathrm{Cl}(\mathfrak{t}^*)$, $\forall a \in \mathcal{A}$, $[c(x), a] = 0$;
2. $\forall x \in \mathrm{Cl}(\mathfrak{t}^*)$, $c(x) \operatorname{Dom} D \subset \operatorname{Dom} D$;
3. $\forall X \in \mathfrak{t}$, $\mu(X) := -\frac{1}{2}[D, c(X^\flat)] - dU(X) \in B(H)$.

Running Example

We can take $c := c^V$, which yields $\mu \equiv 0$.

Vertical geometries

Fix a normalised Ad-invariant inner product $\langle \cdot\,, \cdot \rangle$ for $\mathfrak{k}$.

A *vertical Clifford action* is a G-equivariant $\mathbf{Z}/2$-graded $*$-representation $c : \mathrm{Cl}(\mathfrak{k}^*) \to B(H)$, such that:

1. $\forall x \in \mathrm{Cl}(\mathfrak{k}^*)$, $\forall a \in \mathcal{A}$, $[c(x), a] = 0$;
2. $\forall x \in \mathrm{Cl}(\mathfrak{k}^*)$, $c(x)\,\mathrm{Dom}\,D \subset \mathrm{Dom}\,D$;
3. $\forall X \in \mathfrak{k}$, $\mu(X) := -\frac{1}{2}[D, c(X^\flat)] - dU(X) \in B(H)$.

Running Example

We can take $c := c^V$, which yields $\mu \equiv 0$.

Remarks

1. This generalises to $Z(M(A))^K_{(0)}$-valued inner products on $\mathfrak{k}$.
2. This can be related to Alekseev–Meinrenken's notion of a *connection* for a $\mathfrak{k}$-DGA.

The vertical Dirac operator

The vertical Clifford action c extends (via dU on $\mathfrak{k}$) to a G-equivariant $\mathbb{Z}/2$-graded $*$-representation of $\mathcal{W}(\mathfrak{k})$ by unbounded operators on $H^{\mathrm{alg}} := \bigoplus_{\pi \in \widehat{K}} H_\pi$.

We can therefore define the *vertical Dirac operator* by

$$D_V := c(\mathcal{D}_{\mathfrak{k}}) = c(\varepsilon^i) dU(\varepsilon_i) + \frac{1}{6} \langle \varepsilon_i, [\varepsilon_j, \varepsilon_k] \rangle c(\varepsilon^i \varepsilon^j \varepsilon^k). \quad (3)$$

Running Example

We have $D_V = c^V(\mathcal{D}_{\mathfrak{k}})$ for $\mathcal{D}_{\mathfrak{k}} \in \mathcal{W}(\mathfrak{k}) \subset \mathcal{W}(\mathfrak{g})$.

Remainders & horizontal Dirac operators

A *remainder* for $(\mathcal{A}, H, D; U; c)$ is an K-invariant odd self-adjoint operator $Z \in B(H)$; its *horizontal Dirac operator* is

$$D_h[Z] := D - D_v - Z. \tag{4}$$

Example

In the commutative case (with a generalised Dirac operator & totally geodesic orbits), the *canonical remainder* is

$$Z_c := c(\varepsilon^i)\mu(\varepsilon_i) - \frac{5}{12}\langle \varepsilon_i, [\varepsilon_j, \varepsilon_k]\rangle c(\varepsilon^i \varepsilon^j \varepsilon^k).$$

Running Example

We can take $Z = 0$, which yields $D_h[0] = c^V(\mathcal{D}_{\mathfrak{g},\mathfrak{k}})$.

Strong remainders

Let Z be a remainder for $(\mathcal{A}, H, D; U; c)$, and set

$$\Omega^1_{D-Z,\mathrm{shor}}(\mathcal{A}) := \overline{A \cdot [D - Z, \mathcal{A}^G]}^{B(H)},$$

$$\Omega^1_{D_h[Z],\mathrm{shor}}(\mathrm{Cl}(\mathfrak{k}^*) \otimes \mathcal{A}) := \overline{(\mathrm{Cl}(\mathfrak{k}^*) \otimes A) \cdot [D_h[Z], (\mathrm{Cl}(\mathfrak{k}^*) \otimes \mathcal{A})^K]}^{B(H)}.$$

We say that Z is *strong* if:

$$\forall a \in \mathcal{A}, \quad [D_h[Z], a] \in \Omega^1_{D-Z,\mathrm{shor}}(\mathcal{A}), \tag{5}$$

$$\forall \omega \in \mathrm{Cl}(\mathfrak{k}^*) \otimes \mathcal{A}, \quad [D_h[Z], \omega] \in \Omega^1_{D_h[Z],\mathrm{shor}}(\mathrm{Cl}(\mathfrak{k}^*) \otimes \mathcal{A}). \tag{6}$$

Running example

Our operator $D_h[0] = c^V(\mathcal{D}_{\mathfrak{g},\mathfrak{k}})$ satisfies

$$\forall f \in C^\infty(G), \quad [D_h[0], f] = c(\mathrm{Proj}_{\pi^* T^*(K \backslash G)}\, df) \in \Omega^1_{D,\mathrm{shor}}(C^\infty(G)).$$

Principal K-spectral triples

A *principal K-spectral triple* is $(\mathcal{A}, H, D; U; c)$ with **strong** remainder Z, such that:

1. the K-action α on $A := \overline{\mathcal{A}}^{B(H)}$ is *free*, i.e.,
$$\overline{\operatorname{Span}}\{(k \mapsto \alpha_k(a_1)a_2) \mid a_1, a_2 \in A\} = C(K) \otimes A;$$
2. the K-actions on $\mathrm{Cl}(\mathfrak{k}^*) \otimes A$ and H satisfy
$$\forall \pi \in \widehat{K}, \quad \overline{(\mathrm{Cl}(\mathfrak{k}^*) \otimes A)_\pi \cdot H^K} = H_\pi,$$
$$\{\omega \in \mathrm{Cl}(\mathfrak{k}^*) \otimes A \mid \ \omega|_{H^K} = 0\} = \{0\},$$
$$\overline{(\mathrm{Cl}(\mathfrak{k}^*) \otimes \mathcal{A})^K} = (\mathrm{Cl}(\mathfrak{k}^*) \otimes A)^K.$$

Running example

Condition 1 is simply principality of $G \twoheadrightarrow K\backslash G$; condition 2 follows from tricks with associated vector bundles.

Given a principal K-spectral triple $(\mathcal{A}, H, D; U; c; Z)$:

1. c encodes the vertical (intrinsic) geometry and index theory through

$$(\mathcal{A}, E, S; U^E) := (\mathcal{A}, \overline{\mathrm{Cl}(\mathfrak{k}^*) \otimes A}_{(\mathrm{Cl}(\mathfrak{k}^*)\otimes A)^K}, \mathcal{D}_{\mathfrak{k}}; \mathrm{Ad}^* \otimes \alpha);$$

2. $D^K[Z] := D_h[Z]|_{H^K}$ encodes the horizontal geometry and index theory through $((\mathrm{Cl}(\mathfrak{k}^*) \otimes \mathcal{A})^K, H^K, D^K[Z]; \mathrm{id})$;
3. $[D_h[Z], \cdot]$ encodes vertical extrinsic geometry and the principal connection through

$$[D_h[Z], \cdot] : \mathcal{A} \to \Omega^1_{D-Z,\mathrm{shor}}(\mathcal{A}),$$
$$[D_h[Z], \cdot] : \mathrm{Cl}(\mathfrak{k}^*) \otimes \mathcal{A} \to \Omega^1_{D_h[Z],\mathrm{shor}}(\mathrm{Cl}(\mathfrak{k}^*) \otimes \mathcal{A}).$$

Synthesis

Theorem (Ć.–Mesland)

Let $(\mathcal{A}, H, D; U; c; Z)$ be a principal K-spectral triple. Then:

1. $H \cong E \mathbin{\widehat{\otimes}}_{(\mathrm{Cl}(\mathfrak{k}^*)\otimes A)^K} H^K$ *and* $D_v \cong S \mathbin{\widehat{\otimes}} \mathrm{id}$;
2. $[D_h[Z], \cdot]$ *canonically induces a K-equivariant Hermitian connection* ∇_h *on E, such that* $D_h[Z] \cong \mathrm{id} \mathbin{\widehat{\otimes}}_{\nabla_h} D^K[Z]$;
3. $[D] = [S] \mathbin{\widehat{\otimes}}_{(\mathrm{Cl}(\mathfrak{k}^*)\otimes A)^K} [D^K[Z]]$ *in K-equivariant KK-theory.*

In other words, in K-equivariant unbounded KK-theory,

$$
\begin{aligned}
&(\mathcal{A}, H, D - Z; U) \\
&\quad \cong (\mathcal{A}, E, S; U^E; \nabla) \mathbin{\widehat{\otimes}}_{(\mathrm{Cl}(\mathfrak{k}^*)\otimes\mathcal{A})^K} ((\mathrm{Cl}(\mathfrak{k}^*) \otimes \mathcal{A})^K, H^K, D^K[Z]; \mathrm{id}).
\end{aligned}
$$

Synthesis

Theorem (Ć.–Mesland)

Let $(\mathcal{A}, H, D; U; c; Z)$ be a principal K-spectral triple. Then:

1. $H \cong E \mathbin{\widehat{\otimes}}_{(\mathrm{Cl}(\mathfrak{k}^*)\otimes A)^K} H^K$ *and* $D_v \cong S \mathbin{\widehat{\otimes}} \mathrm{id}$;
2. $[D_h[Z], \cdot]$ *canonically induces a K-equivariant Hermitian connection* ∇_h *on E, such that* $D_h[Z] \cong \mathrm{id} \mathbin{\widehat{\otimes}}_{\nabla_h} D^K[Z]$;
3. $[D] = [S] \mathbin{\widehat{\otimes}}_{(\mathrm{Cl}(\mathfrak{k}^*)\otimes A)^K} [D^K[Z]]$ *in K-equivariant KK-theory.*

In other words, in K-equivariant unbounded KK-theory,

$$(\mathcal{A}, H, D - Z; U) \cong (\mathcal{A}, E, S; U^E; \nabla) \mathbin{\widehat{\otimes}}_{(\mathrm{Cl}(\mathfrak{k}^*)\otimes\mathcal{A})^K} ((\mathrm{Cl}(\mathfrak{k}^*) \otimes \mathcal{A})^K, H^K, D^K[Z]; \mathrm{id}).$$

Remark

The class $[S]$ is the NC wrong-way class for $A \hookleftarrow A^K$.

A word from our sponsors

1. The quantum Weyl algebra $\mathcal{W}(\mathfrak{k})$ defines the natural K-$*$-algebra of vertical NC differential operators.
2. De Commer–Yamashita's proof that a $\mathbf{K}$-C^*-algebra is principal iff it is saturated provides a NC proxy for Gleason's topological slice theorem.
3. Recent work by Kaad–Van Suijlekom and by Van den Dungen allows for maximal generality in the non-unital case.

The only formal bottleneck to working with $\mathbf{K} := K_q$ is point 1.

A word from our sponsors

1. The quantum Weyl algebra $\mathcal{W}(\mathfrak{k})$ defines the natural K-$*$-algebra of vertical NC differential operators.
2. De Commer–Yamashita's proof that a $\mathbf{K}$-C^*-algebra is principal iff it is saturated provides a NC proxy for Gleason's topological slice theorem.
3. Recent work by Kaad–Van Suijlekom and by Van den Dungen allows for maximal generality in the non-unital case.

The only formal bottleneck to working with $\mathbf{K} := K_q$ is point 1.

Plea

Can one construct $\mathcal{W}(\mathfrak{k}_q) \ni \mathcal{D}_{\mathfrak{k}_q}$?

Gauge theory

Gauge comparability

Let $(\mathcal{A}, H, D_0; U; c; 0)$ be a principal K-spectral triple, such that

$$\forall x \in \mathrm{Cl}(\mathfrak{k}^*), \quad [(D_0)_h[0], x] \in \mathrm{Cl}(\mathfrak{k}^*) \cdot \Omega^1_{D_0,\mathrm{shor}}(\mathcal{A}). \tag{7}$$

Let $\mathfrak{D}$ be the set of all D on H making $(\mathcal{A}, H, D; U; c; 0)$ into a principal K-spectral triple satisfying the analogue of (7).

Definition

We say that $D_1, D_2 \in \mathfrak{D}$ are *gauge comparable* if:

1. $\mathrm{Dom}\, D_1 \cap \mathrm{Dom}\, D_2$ is a joint core for D_1 and D_2;
2. $D_1 - D_2 \in B(\mathrm{Dom}\, D_v, H)$;
3. $D_1 - D_2$ supercommutes with $\mathrm{Cl}(\mathfrak{k}^*)$ and $\mathcal{A}^K$.

Denote the gauge comparability class of D_0 by $\mathfrak{At}$ (for Atiyah).

Proposition (Ć.–Mesland)

For any $D_1, D_2 \in \mathfrak{D}$, if D_1 and D_2 are gauge comparable, then

$$[D_1] = [D_2], \quad [D_1^K] = [D_2^K]$$

in K-equivariant KK-theory.

Proof.

Since $D_1 - D_2 \in B(\operatorname{Dom} D_v, H)$, it follows that $D_1^K - D_2^K \in B(H^K)$, so that $[D_1^K] = [D_2^K]$, and hence

$$[D_1] = [S] \mathbin{\widehat{\otimes}}_{(\mathrm{Cl}(\mathfrak{k}^*)\otimes A)^K} [D_1^K] = [S] \mathbin{\widehat{\otimes}}_{(\mathrm{Cl}(\mathfrak{k}^*)\otimes A)^K} [D_2^K] = [D_2]. \qquad \square$$

Proposition (Ć.–Mesland)

For any $D_1, D_2 \in \mathfrak{D}$, if D_1 and D_2 are gauge comparable, then

$$[D_1] = [D_2], \quad [D_1^K] = [D_2^K]$$

in K-equivariant KK-theory.

Proof.

Since $D_1 - D_2 \in B(\operatorname{Dom} D_v, H)$, it follows that $D_1^K - D_2^K \in B(H^K)$, so that $[D_1^K] = [D_2^K]$, and hence

$$[D_1] = [S] \mathbin{\widehat{\otimes}}_{(\mathrm{Cl}(\mathfrak{k}^*)\otimes A)^K} [D_1^K] = [S] \mathbin{\widehat{\otimes}}_{(\mathrm{Cl}(\mathfrak{k}^*)\otimes A)^K} [D_2^K] = [D_2]. \qquad \square$$

Remark

The non-unital version is an honest theorem.

Let's define a *gauge transformation* for $D \in \mathfrak{At}$ to be an even K-invariant unitary $S \in U(H)$, (super)commuting with $\mathrm{Cl}(\mathfrak{k}^*)$ and A^K, such that:

1. $S\mathcal{A}S^* \subset \mathcal{A}$;
2. $S \cdot \operatorname{Dom} D \subset \operatorname{Dom} D$ and $[D, S] \in B(\operatorname{Dom} D_v, H)$;
3. $[D, S]$ supercommutes with $\mathrm{Cl}(\mathfrak{k}^*)$ and $\mathcal{A}^K$.

We define the *gauge group* $\mathfrak{G}$ to be the group of all gauge transformations for one (and hence all!) $D \in \mathfrak{At}$.

The gauge group $\mathfrak{G}$ admits a *gauge action* on $\mathfrak{At}$ by

$$\mathfrak{G} \times \mathfrak{At} \ni (S, D) \mapsto SDS^* \in \mathfrak{At}.$$

Relative gauge potentials

Let's define a *relative gauge potential* for $D \in \mathfrak{At}$ to be an odd K-invariant symmetric operator ω on $\operatorname{Dom} D_v$, such that:

1. $\forall a \in \mathcal{A}$, $[\omega, a] \in \Omega^1_{D,\mathrm{shor}}(\mathcal{A})$;
2. $\omega \in B(\operatorname{Dom} D_v, H)$;
3. ω supercommutes with $\mathrm{Cl}(\mathfrak{k}^*)$ and $\mathcal{A}^K$.

We define the *space of relative gauge potentials* $\mathfrak{at}$ to be the $\mathbf{R}$-vector space of all relative gauge potentials for one (and hence all!) $D \in \mathfrak{At}$.

The gauge group $\mathfrak{G}$ acts (naïvely) on $\mathfrak{at}$ by

$$\mathfrak{G} \times \mathfrak{at} \ni (S, \omega) \mapsto S\omega S^* \in \mathfrak{at}.$$

The affine picture

Theorem (Ć.–Mesland)

1. *The space* $\mathfrak{At}$ *is an affine space modelled on* $\mathfrak{at}$ *with subtraction* $\mathfrak{At} \times \mathfrak{At} \ni (D_1, D_2) \mapsto D_1 - D_2 \in \mathfrak{at}$.
2. *For any fixed* $D \in \mathfrak{At}$*, the homeomorphism*

$$\mathfrak{At} \to \mathfrak{at}, \quad D' \mapsto D' - D$$

intertwines the gauge action of $\mathfrak{G}$ *on* $\mathfrak{At}$ *with*

$$\mathfrak{G} \times \mathfrak{at} \ni (S, \omega) \mapsto S[D, S^*] + S\omega S^* \in \mathfrak{at}.$$

The affine picture

Theorem (Ć.–Mesland)

1. *The space* $\mathfrak{At}$ *is an affine space modelled on* $\mathfrak{at}$ *with subtraction* $\mathfrak{At} \times \mathfrak{At} \ni (D_1, D_2) \mapsto D_1 - D_2 \in \mathfrak{at}$.
2. *For any fixed* $D \in \mathfrak{At}$*, the homeomorphism*

$$\mathfrak{At} \to \mathfrak{at}, \quad D' \mapsto D' - D$$

intertwines the gauge action of $\mathfrak{G}$ *on* $\mathfrak{At}$ *with*

$$\mathfrak{G} \times \mathfrak{at} \ni (S, \omega) \mapsto S[D, S^*] + S\omega S^* \in \mathfrak{at}.$$

Remarks

1. Everything in sight can be suitably topologised.
2. Current technology limits us to the unital case for this!

A concrete *noncommutative* example

Fix $e^{i\theta} \in U(1)$, which generates a $\mathbf{Z}$-action on $U(1)$. Let:

- $\not{D}_{U(1)} := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \frac{d}{dt}$;
- $N : c_c(\mathbf{Z}, \mathbf{C}^2) \to \ell^2(\mathbf{Z}, \mathbf{C}^2)$ be given by

$$N(\delta_n \otimes v) := \delta_n \otimes 2\pi n i \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} v;$$

- $\mathrm{Y} : U(1) \to U(\ell^2(\mathbf{Z}, \mathbf{C}^2))$ be the dual representation;
- $(\mathcal{A}, H; U) := (\mathbf{Z} \ltimes C^\infty(U(1)), \ell^2(\mathbf{Z}, \mathbf{C}^2) \mathbin{\widehat{\otimes}} L^2(U(1), \mathbf{C}^2), \mathrm{Y} \mathbin{\widehat{\otimes}} \mathrm{id}))$;
- $D_0 := N \mathbin{\widehat{\otimes}} \mathrm{id} + \mathrm{id} \mathbin{\widehat{\otimes}} \not{D}_{U(1)}$;
- $c : \mathfrak{u}(1)^* = i\mathbf{R} \ni -i \mapsto \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \mathbin{\widehat{\otimes}} \mathrm{id} \in B(H)$.

Then $(\mathcal{A}, H, D_0; U; c; 0)$ is a principal $U(1)$-spectral triple.

If $\frac{\theta}{2\pi}$ is irrational, then this recovers the irrational NC 2-torus $\mathbf{T}^2_\theta$.

Proposition

We have compatible isomorphisms

$$\{\omega \in \mathfrak{at} \mid \ \omega|_{H^{U(1)}} = 0\} \cong Z^1\left(\mathbf{Z}, \Omega^1_{cts}(U(1), \mathbf{R})\right),$$
$$\{S \in \mathfrak{G} \mid \ S|_{H^{U(1)}} = \mathrm{id}\} \cong Z^1_b\left(\mathbf{Z}, C^\infty(U(1), U(1))\right).$$

Example

For any $\lambda \in \mathbf{R}$, the element $\omega_\lambda \in \mathfrak{at}$ corresponding to

$$(n \mapsto \lambda n \cdot dt) \in Z^1\left(\mathbf{Z}, \Omega^1_{\mathrm{cts}}(U(1), \mathbf{R})\right)$$

yields $D_0 + \omega_\lambda \in \mathfrak{At}$ corresponding to the conformal class of the flat metric on $\mathbf{T}^2_\theta$ parametrized by $\tau = \lambda + i$.