Classical gauge theory on quantum principal bundles

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Shameless self-promotion

B. Ć., *Classical gauge theory on quantum principal bundles*, `arXiv:2108.13789`

B. Ć. and B. Mesland, *Gauge theory on noncommutative Riemannian principal bundles*, Commun. Math. Phys., `arXiv:1912.04179`

Remark

Today, restrict to quantum principal $U_\kappa(1)$-bundles over 2-dimensional NC bases and reformulate definitions to favour transparency over economy out of pure expositional laziness.

We can do everything for structure group H any Hopf $*$-algebra with any bicovariant $*$-FODC and any base while truncating all $*$-DGA at degree 2.

The case study

Irrational noncommutative 2-tori

Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$.

The *smooth* NC *2-torus* $\mathcal{A}_\theta$ is the unital $*$-algebra of rapidly decaying Laurent series in unitaries U_θ, V_θ, such that

$$V_\theta U_\theta = e^{2\pi i \theta} U_\theta V_\theta.$$

$\mathcal{A}_\theta$ is dense and stable under the holomorphic functional calculus in its C^*-completion A_θ, so that

$$K_0(\mathcal{A}_\theta) \cong K_0(A_\theta) \cong \mathbb{Z} + \mathbb{Z}\theta.$$

Define commuting $*$-derivations $\delta_1, \delta_2 : \mathcal{A}_\theta \to \mathcal{A}_\theta$ by

$$\delta_1(U) \coloneqq 2\pi U, \quad \delta_1(V) \coloneqq 0, \quad \delta_2(U) \coloneqq 0, \quad \delta_2(V) \coloneqq 2\pi V.$$

Heisenberg modules

Let $g \in SL(2,\mathbb{Z}) \setminus \{\pm 1\}$, let $\mathcal{E}(g,\theta) := \mathcal{S}(\mathbb{R}) \otimes \mathbb{C}_{g_{21}}$ with

$$(f \cdot U_\theta)(x,k) = \exp\left(2\pi i \left(x - \frac{g_{22}k}{g_{21}}\right)\right) f(x,k),$$

$$(f \cdot V_\theta)(x,k) = f\left(x - \frac{g_{21}\theta + g_{22}}{g_{21}}, k-1\right).$$

Theorem (Connes '80)

1. $\mathcal{E}(g,\theta)$ is FGP, represents $|g_{21}\theta + g_{22}| \in \mathbb{Z} + \theta\mathbb{Z} \cong K_0(A_\theta)$.
2. $\partial_{g,\theta;1}, \partial_{g,\theta;2} : \mathcal{E}(g,\theta) \to \mathcal{E}(g,\theta)$ defined by

$$\partial_{g,\theta;1} f(x,k) := -i f'(x,k), \quad \partial_{g,\theta;2} f(x,k) := 2\pi \frac{g_{21}}{g_{21}\theta + g_{22}} x f(x,k)$$

satisfy $\partial_{g,\theta;i}(f \cdot b) = \partial_{g,\theta;i}(f) \cdot b + f \cdot \delta_i(b)$ for $i = 1, 2$ and

$$[\partial_{g,\theta;1}, \partial_{g,\theta;2}] = -2\pi i \frac{g_{21}}{g_{21}\theta + g_{22}} \mathrm{id}_{\mathcal{E}(g,\theta)}.$$

Heisenberg bimodules

Let $g \triangleright \theta \coloneqq \frac{g_{11}\theta + g_{12}}{g_{21}\theta + g_{22}}$, endow $\mathcal{E}(g, \theta)$ with

$$(U_{g \triangleright \theta} \cdot f)(x, k) = \exp\left(2\pi i \left(\frac{x}{g_{21}\theta + g_{22}} - \frac{k}{g_{21}}\right)\right) f(x, k),$$

$$(V_{g \triangleright \theta} \cdot f)(x, k) = f\left(x - \frac{1}{g_{21}}, k - g_{11}\right).$$

Theorem (Connes '80; Rieffel '81, '83, '88)

$\mathcal{E}(g, \theta)$ is an $(\mathcal{A}_{g \triangleright \theta}, \mathcal{A}_\theta)$-equivalence bimodule.

Question

If $g \triangleright \theta = \theta$, so that θ is *quadratic*, then $\mathcal{E}(g, \theta)$ is a non-trivial NC line bundle on $\mathcal{A}_\theta$.

What is the underlying NC U(1)-gauge theory?

Real quadratic irrationalities

Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$ be quadratic with discriminant Δ, i.e.,

$$\theta = \frac{b + \sqrt{b^2 - 4ac}}{2a}, \quad \Delta \coloneqq b^2 - 4ac$$

for unique $(a, b, c) \in \mathbb{Z}_{\neq 0} \times \mathbb{Z}^2$ with $b^2 - 4ac$ non-square.

Let $(u, v) \in \mathbb{N}^2$ be the *fundamental* solution of $x^2 - y^2\Delta = 4$, i.e.,

$$\forall (s, t) \in \mathbb{N}^2, \quad (s^2 - t^2\Delta = 4) \implies ((s \geqslant u) \wedge (t \geqslant v));$$

the *norm-positive fundamental unit* of $\mathbb{Q}[\theta] = \mathbb{Q}[\Delta]$ is

$$q \coloneqq \frac{u + v\sqrt{\Delta}}{2} \in (\mathbb{Q}[\theta] \setminus \mathbb{Q}) \cap (1, \infty).$$

Line bundles on real multiplication NC tori

Folklore around Pell's equation

Have $\phi : \{g \in SL(2,\mathbb{Z}) : g \triangleright \theta = \theta\} \xrightarrow{\cong} \langle q \rangle \times \{\pm 1\} < \mathbb{Q}[\theta]^\times$, where

$$\phi(g) \coloneqq g_{21}\theta + g_{22}.$$

Get a canonical family of NC line bundles $\{P_n\}_{n\in\mathbb{Z}}$ on $\mathcal{A}_\theta$, where

$$P_n \coloneqq \begin{cases} \mathcal{A}_\theta, & \text{if } n = 0, \\ \mathcal{E}(\phi^{-1}(q^n), \theta), & \text{else.} \end{cases}$$

Idea

Make $P \coloneqq \bigoplus_{n\in\mathbb{Z}} P_n$ into an NC principal U(1)-bundle.

The unital $\mathbb{C}$-algebra

Step 1 (Schwarz '98, Dieng–Schwarz '02, Polishchuk–Schwarz '03)

Make P into a unital algebra, such that for all $m, n \in \mathbb{Z}$

$$\text{multiplication} : P_m \otimes_{\mathcal{A}_\theta} P_n \xrightarrow{\cong} P_{m+n}.$$

For $n \in \mathbb{Z}$, let $\begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \coloneqq \phi^{-1}(q^n)$.

Case 1: if $m = 0$ or $n = 0$, use the $\mathcal{A}_\theta$-bimodule structure.

Case 2: if $m \neq 0$ and $n = -m$, for $f \in P_m$ and $g \in P_n$, set

$$f \cdot g \coloneqq \sum_{j,k \in \mathbb{Z}} U_\theta^j V_\theta^k \sum_{\ell \in \mathbb{Z}_{c_m}} \int_{\mathbb{R}} (V_\theta^{-k} U_\theta^{-j} \cdot f)(q^{-m} x, \ell) g(x, -a_m \ell) dx.$$

Case 3: if $m \neq 0$, $n \neq 0$, $m + n \neq 0$, for $f \in P_m$ and $g \in P_n$, set

$$(f \cdot g)(x, k) \coloneqq \textstyle\sum_{\ell \in \mathbb{Z}_{c_m}} f\Big(\frac{x}{q^m} + q^n \left(\frac{d_{m+n} k}{c_{m+n}} - \frac{\ell}{c_m}\right), a_m d_{m+n} k - \ell\Big) g\Big(x - \left(\frac{d_{m+n} k}{c_{m+n}} - \frac{\ell}{c_n}\right), a_n \ell\Big).$$

The U(1)-$*$-algebra

Step 2 (Polishchuk '04, Vlasenko '06)

Make P into a unital $*$-algebra, such that for all $m \in \mathbb{Z}$,

$$*(P_m) = P_{-m}.$$

Case 1: if $m = 0$, use the $*$-structure on $\mathcal{A}_\theta$.

Case 2: if $m \neq 0$, for $f \in P_m$, set $f^*(x, k) := \overline{f(q^m x, -a_m k)}$.

Step 3

Make P a U(1)-$*$-algebra with spectral subspaces $\{P_n\}_{n\in\mathbb{Z}}$.

For $n \in \mathbb{Z}$, define the U(1)-action on P_n by $\alpha_z(p) := z^n p$.

The principal U(1)-$*$-algebra

Mathematical bricolage

P is a non-trivial principal U(1)-$*$-algebra.

Proof.

P is principal since $P_m \cdot P_n = P_{m+n}$ for all $m, n \in \mathbb{Z}$.

P is non-trivial (i.e., not a crossed product by $\mathbb{Z}$) since,

$$[(P_1)_{\mathcal{A}_\theta}] = [\mathcal{E}(\phi^{-1}(q), \theta)_{\mathcal{A}_\theta}] = q \neq 1 = [(\mathcal{A}_\theta)_{\mathcal{A}_\theta}]$$

in the ordered Abelian group $K_0(\mathcal{A}_\theta) \cong \mathbb{Z} + \mathbb{Z}\theta$. □

Question

Are Connes's constant curvature connections on the P_n induced by a single NC principal connection on P?

Horizontal calculi

The basic calculus

Let $B \coloneqq {}^{\mathrm{co}\,\mathcal{O}(\mathrm{U}(1))}P = P_0 = \mathcal{A}_\theta$.

Let Ω_B be the graded $*$-algebra generated over $B = \Omega_B^0$ by super-central skew-adjoint $\mathrm{d}\tau^1, \mathrm{d}\tau^2 \in \Omega_B^1$.

Define $\mathrm{d}_B : \Omega_B \to \Omega_B$ by $\mathrm{d}_B(\mathrm{d}\tau^1) = \mathrm{d}_B(\mathrm{d}\tau^2) = 0$ and

$$\mathrm{d}_B(b) \coloneqq i\delta_1(b)\,\mathrm{d}\tau^1 + i\delta_2(b)\,\mathrm{d}\tau^2.$$

Then (Ω_B, d_B) is a $*$-differential calculus on $B = \mathcal{A}_\theta$.

Idea

Lift (Ω_B, d_B) to a $\mathrm{U}(1)$-equivariant $*$-DGA $(\Omega_{P,\mathrm{hor}}, \mathrm{d}_{P,\mathrm{hor}})$ on P, such that, for $n \neq 0$,

$$\mathrm{d}_{P,\mathrm{hor}}\big|_{P_n} = i\partial_{\phi^{-1}(q^n),\theta;1}(\cdot)\,\mathrm{d}\tau^1 + i\partial_{\phi^{-1}(q^n),\theta;2}(\cdot)\,\mathrm{d}\tau^2.$$

Horizontal calculi

Definition (cf. Đurđević ’98)

A *horizontal calculus* on P consists of

1. a graded $U(1)$-$*$-algebra $\Omega_{P,\mathrm{hor}}$ over $P = \Omega^0_P$,
2. an extension of $B \hookrightarrow P$ to $\iota : \Omega_B \hookrightarrow \Omega_{P,\mathrm{hor}}$,

such that $\Omega_{P,\mathrm{hor}} = P \cdot \iota(\Omega_B) \cdot P$ and $(\Omega_{P,\mathrm{hor}})^{U(1)} = \iota(\Omega_B)$.

Lemma (Beggs–Majid ’21)

Given that $\Omega_{P,\mathrm{hor}} = P \cdot \iota(\Omega_B) \cdot P$,

$$\left((\Omega_{P,\mathrm{hor}})^{U(1)} = \iota(\Omega_B)\right) \iff \left(\Omega^1_{P,\mathrm{hor}} = P \cdot \iota(\Omega^1_B)\right).$$

Gauge potentials

Let $(\Omega_{P,\text{hor}}, \iota)$ be a horizontal calculus; identify Ω_B with $\iota(\Omega_B)$.

Definition (cf. Đurđević '98)

A *prolongable gauge potential* on P with respect to $\Omega_{P,\text{hor}}$ is a $U(1)$-equivariant $*$-derivation ∇ on $\Omega_{P,\text{hor}}$, such that

$$\nabla|_B = \mathrm{d}_B.$$

We denote by $\mathfrak{At}^{\text{pr}}$ the set of all prolongable gauge potentials.

Since ${}_B(\Omega^1_B)$ is free with basis $\{\mathrm{d}\tau^1, \mathrm{d}\tau^2\}$, so too is ${}_P(\Omega^1_{P,\text{hor}})$.

Hence, each $\nabla \in \mathfrak{At}^{\text{pr}}$ is determined by the unique $U(1)$-equivariant maps $D_1, D_2 : P \to P$, such that

$$\forall p \in P, \quad \nabla(p) = iD_1(p)\mathrm{d}\tau^1 + iD_2(p)\mathrm{d}\tau^2.$$

Horizontal partial derivatives

For $i = 1, 2$, define $\partial_i : P \to P$ by $\partial_i|_B \coloneqq \delta_i$ and, for $n \neq 0$,

$$\partial_i|_{P_n} \coloneqq \partial_{\phi^{-1}(q^n),\theta;i}.$$

Given $\epsilon \in \mathbb{R}^\times$, define $\sigma_\epsilon : P \to P$ by $\sigma_\epsilon|_{P_n} \coloneqq \epsilon^{-n}\,\mathrm{id}_{P_n}$, so that σ_ϵ is a U(1)-equivariant automorphism with

$$\sigma_\epsilon|_B = \mathrm{id}_B, \quad (\sigma_\epsilon \circ *)^2 = \mathrm{id}, \quad \sigma_\epsilon \circ \partial_1 = \partial_1 \circ \sigma_\epsilon, \quad \sigma_\epsilon \circ \partial_2 = \partial_2 \circ \sigma_\epsilon.$$

Proposition (Polishchuk–Schwarz ’03)

The U(1)-equivariant maps $\partial_1, \partial_2 : P \to P$ satisfy, for $i = 1, 2$,

$$\partial_i(p_1 \cdot p_2) = \partial_i(p_1) \cdot \sigma_q(p_2) + p_1 \cdot \partial_i(p_2),$$
$$\partial_i(p^*) = -\sigma_q(\partial_i(p)^*).$$

The horizontal calculus and horizontal covariant derivative

Let $\Omega_{P,\mathrm{hor}}$ be the graded U(1)-$*$-algebra generated over $P = \Omega^0_{P,\mathrm{hor}}$ by skew-adjoint $d\tau^1, d\tau^2 \in (\Omega^1_{P,\mathrm{hor}})^{\mathrm{U}(1)}$ with:

$$\forall p \in P, \quad d\tau^1 \cdot p = \sigma_q(p) \cdot d\tau^1, \quad d\tau^2 \cdot p = \sigma_q(p) \cdot d\tau^2;$$
$$(d\tau^1)^2 = (d\tau^2)^2 = [d\tau^1, d\tau^2] = 0.$$

Then $\Omega_{P,\mathrm{hor}}$ is a horizontal calculus for P with respect to Ω_B.

Proposition

There exists unique $\nabla_0 \in \mathfrak{At}^{\mathrm{pr}}$, such that

$$\forall p \in P, \quad \nabla_0(p) = i\partial_1(p) \cdot d\tau^1 + i\partial_2(p) \cdot d\tau^2.$$

Relative gauge potentials

Definition (cf. Đurđević '98)

A *prolongable relative gauge potential* on P with respect to $\Omega_{\mathrm{P},\mathrm{hor}}$ is a $\mathrm{U}(1)$-equivariant $*$-derivation $\mathbb{A}$ on $\Omega_{\mathrm{P},\mathrm{hor}}$, s.t.

$$\mathbb{A}|_{\Omega_{\mathrm{B}}} = 0.$$

We denote by $\mathfrak{at}^{\mathrm{pr}}$ the set of all prolongable gauge potentials.

Thus, $\mathfrak{At}$ is an $\mathbb{R}$-affine space with space of translations $\mathfrak{at}$.

Proposition

We have an isomorphism $\psi_{\mathfrak{at}^{\mathrm{pr}}} : \mathbb{R}^2 \xrightarrow{\cong} \mathfrak{at}^{\mathrm{pr}}$ given by

$$\psi_{\mathfrak{at}}(s_1, s_2) \coloneqq \left[\mathrm{i}s_1 \mathrm{d}\tau^1 + \mathrm{i}s_2 \mathrm{d}\tau^2, \cdot\right].$$

Gauge transformations

Definition

A *prolongable gauge transformation* of P with respect to $\Omega_{P,\mathrm{hor}}$ is a U(1)-equivariant $*$-automorphism f of $\Omega_{P,\mathrm{hor}}$, s.t.

$$f|_{\Omega_B} = \mathrm{id}_{\Omega_B} .$$

$\mathfrak{G}^{\mathrm{pr}}$ is the group of all prolongable gauge transformations.

Proposition

We have an isomorphism $\psi_{\mathfrak{G}^{\mathrm{pr}}} : U(1) \xrightarrow{\cong} \mathfrak{G}^{\mathrm{pr}}$ given by

$$\psi_{\mathfrak{at}}(z) := \bigoplus_{n \in \mathbb{Z}} z^n \, \mathrm{id}_{(\Omega_{P,\mathrm{hor}})_n} .$$

The gauge action

Proposition

The group $\mathfrak{G}^{\mathrm{pr}}$ *acts* $\mathbb{R}$*-affinely on* $\mathfrak{At}^{\mathrm{pr}}$ *by*

$$f \triangleright \nabla \coloneqq f \circ \nabla \circ f^{-1}.$$

Corollary

In this case, $\mathfrak{G}^{\mathrm{pr}}$ acts trivially on $\mathfrak{At}^{\mathrm{pr}}$.

Remark

Typically (e.g., $P = C^\infty(M) \rtimes_\sigma \mathbb{Z}$ or $P = C^\infty(S^3_\theta)$), the group $\mathfrak{G}^{\mathrm{pr}}$ acts on $\mathfrak{At}^{\mathrm{pr}}$ via a non-trivial homomorphism $\mathfrak{G}^{\mathrm{pr}} \to \mathfrak{at}^{\mathrm{pr}}$.

Total calculi

Preliminaries

Let $\kappa \in (0,\infty)$.

For $n \in \mathbb{Z}$, let $[n]_\kappa \coloneqq n$ if $\kappa = 1$, otherwise $[n]_\kappa \coloneqq \frac{\kappa^n - 1}{\kappa - 1}$.

Let $\Lambda_\kappa \coloneqq \mathbb{C}[\mathrm{d}_\kappa t]/\langle(\mathrm{d}_\kappa t)^2\rangle$ with $\mathrm{d}_\kappa t$ skew-adjoint.

Given a graded $\mathrm{U}(1)$-$*$-algebra Ω, let

$$\Lambda_\kappa \rtimes \Omega \coloneqq \Lambda_\kappa \mathbin{\widehat{\otimes}} \Omega / \langle \mathrm{d}_\kappa t \mathbin{\widehat{\otimes}} \omega - (-1)^{|\omega|}\sigma_\kappa(\omega) \mathbin{\widehat{\otimes}} \mathrm{d}_\kappa t : \omega \in \Omega \rangle$$

with $\mathrm{d}_\kappa t$ defined to be $\mathrm{U}(1)$-invariant.

Definition (cf. Đurđević ’97)

The *vertical calculus* of P w.r.t. $\mathrm{U}_\kappa(1)$ is the $\mathrm{U}(1)$-equivariant $*$-DGA $(\Omega_{P,\mathrm{ver};\kappa}, \mathrm{d}_{P,\mathrm{ver};\kappa})$, where $\Omega_{P,\mathrm{ver};\kappa} \coloneqq \Lambda_\kappa \rtimes P$ and

$$\forall n \in \mathbb{Z}, \forall p \in P_n, \quad \mathrm{d}_{P,\mathrm{ver};\kappa}(p) \coloneqq \mathrm{d}_\kappa t \cdot 2\pi i [n]_\kappa p.$$

Quantum principal $\mathrm{U}_\kappa(1)$-bundles

Definition (cf. Brzeziński–Majid '93, Đurđević '97, Beggs–Majid '21)

A $\mathrm{U}(1)$-equivariant $*$-DGA (Ω_P, d_P) on P is κ-*principal* over (Ω_B, d_B) iff $\Omega_B \hookrightarrow (\Omega_P)^{\mathrm{U}(1)}$ and $\mathrm{d}_P|_{\Omega_B} = \mathrm{d}_B$, and

1. we can define surjective $\mathrm{ver}_\kappa[\mathrm{d}_P] : \Omega_P \to \Omega_{P,\mathrm{ver};\kappa}$ by
$$\mathrm{ver}_\kappa[\mathrm{d}_P]|_P = \mathrm{id}_P, \quad \mathrm{ver}_\kappa[\mathrm{d}_P] \circ \mathrm{d}P|_P = \mathrm{d}_{P,\mathrm{ver};\kappa},$$
and $\ker \mathrm{ver}_\kappa[\mathrm{d}_P] = \Omega_P \cdot \Omega^1_B$;
2. we can define $\mathrm{int}_\kappa[\mathrm{d}_P] : \Omega_P \to \Lambda_\kappa \rtimes \Omega_P$ by
$$\mathrm{int}_\kappa[\mathrm{d}_P]|_P = \mathrm{id}_P, \quad \mathrm{int}_\kappa[\mathrm{d}_P]|_{\Omega^1_P} = \mathrm{id}_{\Omega^1_P} + \mathrm{ver}_\kappa[\mathrm{d}_P]|_{\Omega^1_P},$$
and $\ker(\mathrm{id} - \mathrm{int}_\kappa[\mathrm{d}_P]) = P \cdot \Omega_B$ is a horizontal calculus.

Synthesis of total calculi

We have 2-dim'l (Ω_B, d_B) on B and compatible $\Omega_{P,\mathrm{hor}}$ on P.

Definition

We say that $\nabla \in \mathfrak{At}^{\mathrm{pr}}$ is κ-*adapted* whenever there exists $F[\nabla] : \Omega^1_{P,\mathrm{ver};\kappa} \to \Omega^2_{P,\mathrm{hor}}$, the *curvature 2-form* of ∇, such that

$$\nabla^2\big|_P = \mathrm{i}F[\nabla] \circ d_{P,\mathrm{ver};\kappa}|_P \,.$$

Let $\mathfrak{At}^{\mathrm{pr}}_\kappa$ denote the quadric subset of all κ-adapted $\nabla \in \mathfrak{At}^{\mathrm{pr}}$.

Theorem

We have $\mathfrak{At}^{\mathrm{pr}}_\kappa \neq \emptyset$ iff $\kappa = q^2$, in which case $\mathfrak{At}^{\mathrm{pr}}_{q^2} = \mathfrak{At}^{\mathrm{pr}}$ and

$$\forall \nabla \in \mathfrak{At}^{\mathrm{pr}} = \mathfrak{At}^{\mathrm{pr}}_{q^2}, \quad F[\nabla](d_{q^2} t) = -\mathrm{i} q c_1 d\tau^1 \wedge d\tau^2.$$

Synthesis of total calculi

Proposition (cf. Đurđević ’10)

Let $\nabla \in \mathfrak{At}^{\mathrm{pr}}$. *Then* ∇ *is* κ*-adapted iff the map*

$$P \to \Omega^1_{P,\mathrm{ver};\kappa} \oplus \Omega^1_{P,\mathrm{hor}}, \quad p \mapsto \mathrm{d}_{P,\mathrm{ver};\kappa}(p) + \nabla(p)$$

extends to a $\mathrm{U}(1)$*-equvariant* $*$*-derivation* $\mathrm{d}_{P,\nabla;\kappa}$ *on*

$$\Omega_{P,\oplus;\kappa} \coloneqq \Lambda_\kappa \rtimes \Omega_{P,\mathrm{hor}},$$

such that $(\Omega_{P,\oplus;\kappa}, \mathrm{d}_{P,\nabla;\kappa})$ *is an* κ*-principal* $*$*-DGA on* P *with*

$$\mathrm{ver}[\mathrm{d}_{P,\nabla;\kappa}]|_{\Omega^1_{P,\oplus;\kappa}} = \mathrm{Proj}_{\Omega^1_{P,\mathrm{ver};\kappa}}.$$

Consequence

We must view P as a quantum principal $\mathrm{U}_{q^2}(1)$-bundle!

Prolongable connections

Definition (cf. Brzeziński–Majid '93, Hajac '96, Đurđević '97)

Let (Ω_P, d_P) be an κ-principal $*$-DGA for P over (Ω_B, d_B). A *prolongable connection* on (Ω_P, d_P) is a $U(1)$-equivariant endomorphism Π of ${}_P(\Omega^1_P)_P$, such that:

1. id_P extends via Π to $\mathrm{ver}_\Pi : \Omega_P \to \Omega_P$, such that
$$(\mathrm{ver}_\Pi)^2 = \mathrm{ver}_\Pi, \quad \ker \mathrm{ver}_\Pi = \ker \mathrm{ver}_\kappa[d_P];$$
2. id_P extends via $\mathrm{id}_{\Omega^1_P} - \Pi$ to $\mathrm{hor}_\Pi : \Omega_P \to \Omega_P$, such that
$$(\mathrm{hor}_\Pi)^2 = \mathrm{hor}_\Pi, \quad \operatorname{ran} \mathrm{hor}_\Pi = \ker \mathrm{int}_\kappa[d_P].$$

Example

Given $\nabla \in \mathfrak{At}^{\mathrm{pr}}$, we have $\Pi_\oplus := \mathrm{Proj}_{\Omega^1_{P,\mathrm{ver};\kappa}}$ on $(\Omega_{P,\oplus;\kappa}, d_{P,\nabla;\kappa})$.

The abstract gauge groupoid

Definition

The *prolongable abstract gauge groupoid* of P with respect to $U_\kappa(1)$ and $\Omega_{P,\mathrm{hor}}$ is the groupoid $\mathcal{G}_\kappa$ defined as follows:

1. an object is a $U_\kappa(1)$-principal $*$-DGA (Ω_P, d_P) for P over (Ω_B, d_B) admitting a prolongable connection, such that
$$\ker \mathrm{int}_\kappa[d_P] = \Omega_{P,\mathrm{hor}};$$
2. an arrow from (Ω_1, d_1) to (Ω_2, d_2) is $U(1)$-equivariant $f : \Omega_1 \xrightarrow{\cong} \Omega_2$, such that $f|_{\Omega_B} = \mathrm{id}_{\Omega_B}$ and
$$f \circ d_1 = d_2 \circ f, \quad (\mathrm{id}_{\Lambda^1_H} \otimes f) \circ \mathrm{int}_\kappa[d_1] = \mathrm{int}_\kappa[d_2] \circ f;$$
3. the groupoid law is composition of maps.

A reconstruction theorem

Theorem

Let $\mathcal{A}_\kappa$ be the set of all triples $(\Omega_P, d_P; \Pi)$, where (Ω_P, d_P) is an object in $\mathcal{G}_\kappa$ and Π is a prolongable connection on (Ω_P, d_P). Then $\Sigma_\kappa : \mathfrak{G}^{\mathrm{pr}} \ltimes \mathfrak{At}_\kappa^{\mathrm{pr}} \to \mathcal{G}_\kappa \ltimes \mathcal{A}_\kappa$ given by

$$(f, \nabla) \mapsto (f : (\Omega_{P,\oplus;\kappa}, d_{P,\nabla;\kappa}; \Pi_\oplus) \mapsto (\Omega_{P,\oplus;\kappa}, d_{P,f \triangleright \nabla;\kappa}; \Pi_\oplus))$$

is an equivalence of groupoids with explicit homotopy inverse.

Corollary

In our case,

$$\mathcal{G}_\kappa \ltimes \mathcal{A}_\kappa \simeq \mathfrak{G}^{\mathrm{pr}} \ltimes \mathfrak{At}_\kappa^{\mathrm{pr}} \cong \begin{cases} \mathrm{U}(1) \ltimes_{\mathrm{trivial}} \mathbb{R}^2, & \text{if } \kappa = q^2, \\ \emptyset, & \text{else.} \end{cases}$$

Isomorphism of total calculi

Definition

We say that $\mathbb{A} \in \mathfrak{at}^{\mathrm{pr}}$ is *canonically* κ*-adapted* whenever there exists $\omega[\mathbb{A}] : \Omega^1_{P,\mathrm{ver};\kappa} \to \Omega^1_{P,\mathrm{hor}}$, the *relative connection* 1*-form* of $\mathbb{A}$, such that

$$\mathbb{A}|_P = \omega[\nabla] \circ \mathrm{d}_{P,\mathrm{ver};\kappa}|_P\,, \quad \omega[\nabla](\mathrm{d}_\kappa t)^2 = 0.$$

Let $\mathfrak{at}^{\mathrm{pr}}_{\mathrm{can},\kappa}$ be the subset of all canonically κ-adapted $\mathbb{A} \in \mathfrak{at}^{\mathrm{pr}}$.

Theorem

We have $\mathfrak{at}^{\mathrm{pr}}_{\mathrm{can},\kappa} \neq 0$ iff $\kappa = q$, in which case $\mathfrak{at}^{\mathrm{pr}}_{\mathrm{can},q} = \mathfrak{at}^{\mathrm{pr}}$ and

$$\forall (s_1, s_2) \in \mathbb{R}^2, \quad \omega\left[\psi_{\mathfrak{at}^{\mathrm{pr}}}(s_1, s_2)\right](\mathrm{d}_q t) = -\tfrac{q-1}{2\pi}(s_1 \mathrm{d}\tau^1 + s_2 \mathrm{d}\tau^2).$$

Isomorphism of total calculi

Let $\mu_\kappa : \mathcal{G}_\kappa \to \mathrm{Aut}(P)$ be the forgetful map, so that (Ω_1, d_1), $(\Omega_2, d_2) \in \mathrm{Obj}(\mathcal{G}_\kappa)$ are isomorphic over id_P as U(1)-equivariant $*$-DGA over P iff $[(\Omega_1, d_1)] = [(\Omega_2, d_2)]$ in $\mathcal{G}_\kappa / \ker \mu_\kappa$.

Theorem

The subset $\mathfrak{at}^{\mathrm{pr}}_{\mathrm{can},\kappa}$ is a $\mathfrak{G}^{\mathrm{pr}}$-invariant subspace of $\mathfrak{at}^{\mathrm{pr}}$ that leaves $\mathfrak{At}^{\mathrm{pr}}_\kappa$ invariant under translation, and Σ_κ descends to

$$\mathfrak{G}^{\mathrm{pr}} \ltimes \mathfrak{At}^{\mathrm{pr}}_\kappa / \mathfrak{at}^{\mathrm{pr}}_{\mathrm{can},\kappa} \xrightarrow{\cong} \mathcal{G}_\kappa / \ker \mu_\kappa.$$

Corollary

In our case, since $\mathfrak{At}^{\mathrm{pr}}_{\mathbf{q}^2} = \mathfrak{At}^{\mathrm{pr}}$ and $\mathfrak{at}^{\mathrm{pr}}_{\mathrm{can},\mathbf{q}^2} = 0$,

$$\mathcal{G}_{\mathbf{q}^2} / \ker \mu_{\mathbf{q}^2} \cong \mathfrak{G}^{\mathrm{pr}} \ltimes \mathfrak{At}^{\mathrm{pr}} \cong U(1) \ltimes_{\mathrm{trivial}} \mathbb{R}^2.$$