ERRATUM: "A RECONSTRUCTION THEOREM FOR CONNES–LANDI DEFORMATIONS OF COMMUTATIVE SPECTRAL TRIPLES"

BRANIMIR ĆAĆIĆ

Abstract. We strengthen the orientability condition in our definition of θ-commutative spectral triple to resolve an issue with the proof of our main theorem. In particular, we show that this corrected condition is still satisfied in the relevant commutative case.

In [1], we extended Connes's reconstruction theorem for commutative spectral triples [2] to Connes–Landi deformations of commutative spectral triples. This involved suitably generalizing Connes's notion of commutative spectral triple to the relevant equivariant case [1, Def. 4.5] and proving explicit compatibility of this definition with Connes–Landi deformation [1, Thm 4.10]. However, our orientability condition [1, Def. 4.5.(3)] is too weak for the relevant part of our compatibility proof [1, §4.2.2]. In this erratum, we strengthen the orientability condition as needed and check that it remains consistent with the relevant commutative case.

From now on, we fix a compact Abelian Lie group G with Pontrjagin dual $\widehat{G}$, and we reprise the notation and terminology of [1]. Moreover, given a Fréchet G-$*$-algebra $\mathcal{A}$, we set $\mathcal{A}^{\mathrm{fin}} := \bigoplus^{\mathrm{alg}}_{\mathbf{x}\in\widehat{G}} \mathcal{A}_{\mathbf{x}}$.

We now explain the problem with our orientability condition—and its resolution. Let $\mathcal{A}$ be a nuclear Fréchet G-$*$-algebra, $p \in \mathbb{N}$, and $\Theta : \hat{G} \times \hat{G} \to \mathbb{T}$ a bicharacter representing $\theta \in H^2(\hat{G}, \mathbb{T})$. To prove stability of orientability under Connes–Landi deformation by θ, we would like to use a map $\Theta_* : \mathcal{A}^{\otimes(p+1)} \to \mathcal{A}_\Theta^{\otimes(p+1)}$ that satisfies

$$\Theta_*(a_0 \otimes a_1 \otimes \cdots \otimes a_p) := \exp\left(2\pi \mathrm{i} \sum_{1\leq j<k\leq p} \Theta(\mathbf{x}_j, \mathbf{x}_k)\right) a_0 \otimes a_1 \otimes \cdots \otimes a_p \tag{1}$$

for all $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_p \in \widehat{G}$ and $a_0 \in \mathcal{A}_{\mathbf{x}_0}$, $a_1 \in \mathcal{A}_{\mathbf{x}_1}, \ldots,$ $a_p \in \mathcal{A}_{\mathbf{x}_p}$. Despite the claim of [1, Lemma 4.15], this will generally define a map from the *algebraic* tensor product $\mathcal{A}^{\otimes(p+1)}$ to the *topological* tensor product $\mathcal{A}_\Theta^{\widehat{\otimes}(p+1)}$. Nonetheless, we do obtain a map $\Theta_* : (\mathcal{A}^{\mathrm{fin}})^{\otimes(p+1)} \to (\mathcal{A}^{\mathrm{fin}}_\Theta)^{\otimes(p+1)}$. This motivates us to correct [1, Def. 4.5.(3)] by replacing $\mathcal{A}^{\otimes(p+1)}$ by $(\mathcal{A}^{\mathrm{fin}})^{\otimes(p+1)}$ wherever it appears.

Correction. In [1, Def. 4.5], we correct condition (3) to read as follows.

(3) *Orientability:* Define $\epsilon_\theta : (\mathcal{A}^{\mathrm{fin}})^{\otimes(p+1)} \to (\mathcal{A}^{\mathrm{fin}}_\Theta)^{\otimes(p+1)}$ by setting

$$\epsilon_\theta(a_0 \otimes a_1 \otimes \cdots \otimes a_p)$$
$$:= \frac{1}{p!} \sum_{\sigma\in S_p} \exp\left(2\pi\mathrm{i} \sum_{\substack{j<k \\ \sigma(j)>\sigma(k)}} \iota(\theta)(\mathbf{x}_j, \mathbf{x}_k)\right) (-1)^\sigma a_0 \otimes a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(p)}$$

for all $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_p \in \widehat{G}$ and $a_0 \in \mathcal{A}_{\mathbf{x}_0}$, $a_1 \in \mathcal{A}_{\mathbf{x}_1}, \dots, a_p \in \mathcal{A}_{\mathbf{x}_p}$, and say that $\mathbf{c} \in \mathcal{A} \otimes (\mathcal{A}^{\mathrm{fin}})^{\otimes p}$ is *θ-antisymmetric* if $\epsilon_\theta(\mathbf{c}) = \mathbf{c}$. There exists a G-invariant θ-antisymmetric $\mathbf{c} \in (\mathcal{A}^{\mathrm{fin}})^{\otimes(p+1)}$ such that $\chi \coloneqq \pi_D(\mathbf{c})$ is a self-adjoint unitary satisfying, for all $a \in \mathcal{A}$, the equations

$$L(a)\chi = \chi L(a), \qquad [D, L(a)]\chi = (-1)^{p+1}\chi[D, L(a)].$$

We can now straightforwardly correct [1, §4.2.2] by replacing every instance of $\mathcal{A}^{\otimes(p+1)}$ and $\mathcal{A}_\Theta^{\otimes(p+1)}$ by $(\mathcal{A}^{\mathrm{fin}})^{\otimes(p+1)}$ and $(\mathcal{A}_\Theta^{\mathrm{fin}})^{\otimes(p+1)}$, respectively. The only non-trivial point to check is that a G-equivariant concrete commutative spectral triple is still 0-commutative with respect to the corrected definition.

Proposition. *Let (X, g) be a p-dimensional compact oriented Riemannian G-manifold, $E \to X$ a G-equivariant Hermitian vector bundle, and D a G-invariant self-adjoint Dirac-type operator on E. Then $(C^\infty(X), L^2(X, E), D)$ defines a p-dimensional 0-commutative spectral triple with respect to the corrected definition.*

Lemma 1 ([1, Proof of Lemma 2.33])**.** *Let $\mathcal{A}$ be a Fréchet pre-G-C^*-algebra, $\mathcal{E}$ a Hermitian f.g.p. G-$\mathcal{A}$-module, and $\{\xi_1, \dots, \xi_n\} \subset \mathcal{E}$ a finite algebraic generating set for the right $\mathcal{A}$-module $\mathcal{E}$. If $\{\eta_1, \dots, \eta_n\} \subset \mathcal{E}$ satisfies $\|\xi_i - \eta_i\| < \frac{1}{n}$ for each $i \in \{1, \dots, n\}$, then it algebraically generates the right $\mathcal{A}$-module $\mathcal{E}$.*

Lemma 2. *Let X be a compact G-manifold. There exist $\mathbf{x}_1, \dots, \mathbf{x}_m \in \widehat{G}$ and non-zero $b_1 \in C^\infty(X)_{\mathbf{x}_1}, \dots, b_m \in C^\infty(X)_{\mathbf{x}_m}$ such that $\{\mathrm{d}b_1, \dots, \mathrm{d}b_m\}$ algebraically generates the $C^\infty(X)$-module $\Omega^1(X)$.*

Proof. First, using a finite atlas for X together with a subordinate smooth partition of unity, construct $a_1, \dots, a_n \in C^\infty(X)$ such that $\{\mathrm{d}a_1, \dots, \mathrm{d}a_n\}$ algebraically generates the $C^\infty(X)$-module $\Omega^1(X)$. Next, fix a G-invariant Riemannian metric g on X, thereby making $\Omega^1(X)$ into a Hermitian f.g.p. G-$C^\infty(X)$-module suitably topologized by a countable family of norms including the pre-Hilbert $C^\infty(X)$-module norm $\|\cdot\|$ on $\Omega^1(X)$ induced by g, see [1, §§2.2, 2.4]. Now, for each $i \in \{1, \dots, n\}$, since the Fourier expansion $\mathrm{d}a_i = \sum_{\mathbf{x} \in \widehat{G}} \widehat{\mathrm{d}a_i}(\mathbf{x})$ is, in particular, absolutely convergent with respect to $\|\cdot\|$, there exist finite $F_i \subset \widehat{G}$ such that $\left\|\mathrm{d}a_i - \sum_{\mathbf{x} \in F_i} \widehat{\mathrm{d}a_i}(\mathbf{x})\right\| < \frac{1}{n}$. Thus, by G-equivariance and Fréchet-continuity of $\mathrm{d} : C^\infty(X) \to \Omega^1(X)$, it follows, for each $i \in \{1, \dots, n\}$, that $\mathrm{d}a_i = \mathrm{d}a_i'$, where $a_i' \coloneqq \sum_{\mathbf{x} \in F_i} \widehat{a_i}(\mathbf{x}) \in C^\infty(X)^{\mathrm{fin}}$. By Lemma 1, it now follows that $\{\mathrm{d}a_1', \dots, \mathrm{d}a_m'\}$ algebraically generates the $C^\infty(X)$-module $\Omega^1(X)$. Finally, take $\{b_1, \dots, b_m\}$ to be the distinct non-zero elements of the finite set $\{\widehat{a_i}(\mathbf{x}) : 1 \leq i \leq n, \mathbf{x} \in \widehat{G}\}$, whence, for each $k \in \{1, \dots, m\}$, we determine $\mathbf{x}_k \in \widehat{G}$ by the inclusion $b_k \in C^\infty(X)_{\mathbf{x}_k}$. □

Proof of the Proposition. The only remaining issue is proving that the corrected orientability condition is satisfied. Let $\star$ be the G-invariant Hodge star operator on (M, g) with its given orientation. Let $\pi_\wedge : C^\infty(X)^{\otimes(p+1)} \to \Omega^p(X)$ denote the surjective G-equivariant left $C^\infty(X)$-linear map given by

$$\pi_\wedge(c_0 \otimes c_1 \otimes \cdots \otimes c_p) \coloneqq c_0 \cdot \mathrm{d}c_1 \wedge \cdots \wedge \mathrm{d}c_p$$

for all $c_0, c_1, \dots, c_p \in C^\infty(X)$. By [2, Proof of Thm 11.4], *mutatis mutandis*, it therefore suffices to find G-invariant 0-antisymmetric $\mathbf{c} \in (C^\infty(X)^{\mathrm{fin}})^{\otimes(p+1)}$ such that $\pi_\wedge(\mathbf{c}) = \star(1)$.

By Lemma 2, there exist characters $\mathbf{x}_1, \ldots, \mathbf{x}_m \in \widehat{G}$ and non-zero isotypical vectors $b_1 \in C^\infty(X)_{\mathbf{x}_1}, \ldots, b_m \in C^\infty(X)_{\mathbf{x}_m}$ such that $\{\mathrm{d}b_1, \ldots, \mathrm{d}b_m\}$ algebraically generates the $C^\infty(X)$-module $\Omega^1(X)$. Given $i_1, \ldots, i_p \in \{1, \ldots, m\}$, since

$$\star(\mathrm{d}b_{i_1} \wedge \cdots \wedge \mathrm{d}b_{i_p})(t \cdot x) = \exp\left(2\pi\mathrm{i} \sum_{k=1}^{p} \langle \mathbf{x}_{i_k}, t\rangle\right) \star (\mathrm{d}b_{i_1} \wedge \cdots \wedge \mathrm{d}b_{i_p})(x)$$

for all $x \in X$, and $t \in G$, it follows that the zero locus of $\mathrm{d}b_{i_1} \wedge \cdots \wedge \mathrm{d}b_{i_p} \in \Omega^p(X)$ is a G-invariant subset of X. Thus, the compact G-manifold X admits a finite cover $\{U_1, \ldots, U_q\}$ by G-invariant open sets, where, for each $1 \leq j \leq q$, there exist indices $1 \leq i_{j;1} < \cdots < i_{j;p} \leq m$ such that $\mathrm{d}b_{i_{j;1}} \wedge \cdots \wedge \mathrm{d}b_{i_{j;p}}$ is nowhere vanishing on U_j; for convenience, set $\mathbf{y}_j \coloneqq \sum_{k=1}^{p} \mathbf{x}_{i_{j;k}}$ for each $j \in \{1, \ldots, q\}$. Finally, fix a G-invariant smooth partition of unity $\{\rho_1, \ldots, \rho_q\}$ for X subordinate to $\{U_1, \ldots, U_q\}$, and observe that $\star(1) = \sum_{j=1}^{q} a_j \cdot \mathrm{d}b_{i_{j;1}} \wedge \cdots \wedge \mathrm{d}b_{i_{j;p}}$, where

$$a_j \coloneqq \rho_j \cdot \star(\mathrm{d}b_{i_{j;1}} \wedge \cdots \wedge \mathrm{d}b_{i_{j;p}})^{-1} \in C_c^\infty(U_j) \cap C^\infty(X)_{-\mathbf{y}_j}.$$

for each $j \in \{1, \ldots, q\}$. Thus, at last, we may simply take

$$\mathbf{c} \coloneqq \sum_{j=1}^{q} \frac{1}{p!} \sum_{\sigma \in S_p} (-1)^\sigma a_j \otimes b_{i_{j;\sigma(1)}} \otimes \cdots \otimes b_{i_{j;\sigma(p)}}. \qquad \square$$

References

[1] Branimir Ćaćić, *A reconstruction theorem for Connes-Landi deformations of commutative spectral triples*, J. Geom. Phys. **98** (2015), 82–109.
[2] Alain Connes, *On the spectral characterization of manifolds*, J. Noncommut. Geom. **7** (2013), no. 1, 1–82.

Department of Mathematics & Statistics, University of New Brunswick, PO Box 4400, Fredericton, NB, E3B 5A3, Canada
Email address: bcacic@unb.ca