# Differentiable Cuntz–Pimsner constructions
## Variations on a theme

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# Outline

Abstract Cuntz–Pimsner in a 2-group

Differentiable Cuntz–Pimsner for horizontal calculi

Differentiable Cuntz–Pimsner for total calculi

Lifting representations of first-order differential calculi

# Coherent 2-groups

## Definition (Sinh '75, Laplaza '83, Baez–Lauda '04)

A *coherent 2-group* G consists of:

1. a monoidal category $(G, \otimes, \alpha, 1_G, \lambda, \rho)$ where every arrow is invertible;
2. a function $(g \mapsto \overline{g}) : \mathrm{Ob}(G) \longrightarrow \mathrm{Ob}(G)$ called *monoidal inversion*;
3. a family of arrows $\mathrm{ev} = \{\mathrm{ev}_g : \overline{g} \otimes g \longrightarrow 1_G\}_{g \in \mathrm{Ob}(G)}$ called *evaluation*.

## Example

A group $\Gamma$ defines a coherent 2-group: take the discrete category on $\Gamma$ with the strict monoidal structure given by the group law, monoidal inversion given by inversion, and evaluation given by the group law.

## Theorem (Laplaza '83)

*Let* G *be a coherent 2-group. Monoidal inversion uniquely extends to a functor* $G \longrightarrow G$ *that makes evaluation into a natural isomorphism.*

# Coherence

## Definition

An arrow in a coherent 2-group G is *elementary* if it can be constructed via finitely many applications of composition, inversion, monoidal inversion, and $\otimes$ from

$$\{\mathrm{id}_g, \lambda_g, \rho_g, \mathrm{ev}_g \mid g \in \mathrm{Ob}(G)\} \cup \{\alpha_{g,h,k} \mid g, h, k \in \mathrm{Ob}(G)\}.$$

## Theorem (Ulbrich '81, Laplaza '83)

*Let G be a coherent 2-group. For every $(g, h) \in \mathrm{Ob}(G)^2$, there is at most one elementary arrow $u : g \to h$ in G.*

## Corollary

*Let G be a coherent 2-group. There exist unique elementary $\star : 1_G \to \overline{1_G}$ and elementary natural isomorphisms*

$$\mathrm{bb} = \{\mathrm{bb}_g : g \to \overline{\overline{g}}\}_{g \in \mathrm{Ob}(G)}, \quad \Upsilon = \{\Upsilon_{g,h} : \overline{g \otimes h} \to \overline{h} \otimes \overline{g}\}_{(g,h) \in \mathrm{Ob}(G)}$$

*that make G into a strong bar category à la Beggs–Majid.*

# Hermitian line modules

## Definition (cf. Beggs–Brzeziński ’14, Arici–Kaad–Landi ’16)

Let $B$ be a unital pre-$C^*$-algebra. A *Hermitian line $B$-module* is a right pre-Hilbert $B$-module $E$ equipped with a $*$-isomorphism $B \to \mathrm{End}_B^*(E)$, such that:

$$\exists e_1, \dots, e_m \in E,\ \forall e \in E, \qquad e = \sum_{i=1}^{m} e_i \cdot (e_i, e_i),$$

$$\exists \epsilon_1, \dots, \epsilon_n \in E, \qquad 1 = \sum_{j=1}^{n} (\epsilon_j, \epsilon_j).$$

Its *conjugate* is the Hermitian line module $\overline{E} \coloneqq \{\overline{e} \mid e \in E\}$ on $B$ with

$$\forall e \in E,\ \forall b_1, b_2 \in B, \qquad b_1 \cdot \overline{e} \cdot b_2 \coloneqq \overline{b_2^* \cdot e \cdot b_1^*},$$

$$\forall e_1, e_2, e \in E, \qquad (\overline{e_1}, \overline{e_2}) \cdot e \coloneqq e_1 \cdot (e_2, e).$$

# The Picard 2-group

## Definition (cf. Fröhlich '74, Beggs–Majid '09, Beggs–Brzeziński '14)

The *Picard 2-group* of a unital pre-$C^*$-algebra $B$ is the coherent 2-group $\mathrm{Pic}(B)$ defined as follows:

1. an object is a Hermitian line $B$-module $E$;
2. an arrow $u : E \longrightarrow F$ is an isomorphism of $B$-bimodules, such that
$$\forall e_1, e_2 \in E, \qquad (u(e_1), u(e_2)) = (e_1, e_2);$$
3. a tensor product $E \otimes F$ is the $B$-bimodule $E \otimes_B F$ with inner product
$$\forall e_1, e_2 \in E, \ \forall f_1, f_2 \in F, \qquad (e_1 \otimes f_1, e_2 \otimes f_2) \coloneqq (e_1, (f_1, f_2) \cdot e_2);$$
4. the inverse of an object $E$ is its conjugate $\overline{E}$, with
$$\forall e_1, e_2 \in E, \qquad \mathrm{ev}_E(\overline{e_1} \otimes e_2) \coloneqq (e_1, e_2).$$

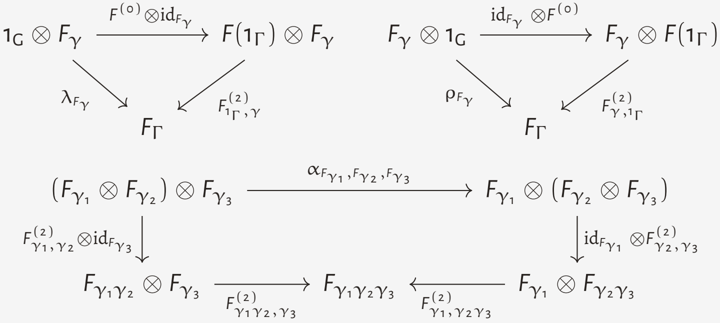
# Homomorphisms from groups to 2-groups

## Definition (cf. Baez–Lauda '04)

Let $\Gamma$ be a group and G a coherent 2-group. A *homomorphism* $F : \Gamma \to \mathrm{G}$ comprises a function $F : \Gamma \to \mathrm{Ob}(\mathrm{G})$, an arrow $F^{(0)} : 1_{\mathrm{G}} \to F_{1_\Gamma}$, and a family of arrows

$$\{F^{(2)}_{\gamma_1,\gamma_2} : F_{\gamma_1} \otimes F_{\gamma_2} \to F_{\gamma_1\gamma_2}\}_{(\gamma_1,\gamma_2)\in\Gamma^2},$$

such that the following diagrams commute for all $\gamma, \gamma_1, \gamma_2, \gamma_3 \in \Gamma$:



# Bar homomorphisms

## Definition (Beggs–Majid ’09)

Let $\Gamma$ be a group and let $\mathrm{G}$ be a coherent 2-group. A *bar homomorphism* $F : \Gamma \to \mathrm{G}$ is a homomorphism $F : \Gamma \to \mathrm{G}$ equipped with $\{F_\gamma^{(-1)} : \overline{F_\gamma} \to F_{\gamma^{-1}}\}_{\gamma\in\Gamma}$ making the following commute for all $\gamma, \gamma_1, \gamma_2 \in \Gamma$:

$$
\begin{array}{ccc}
\overline{1_{\mathrm{G}}} & \xrightarrow{\overline{F^{(0)}}} & \overline{F_{1_\Gamma}} \\
\star\big\uparrow & & \big\downarrow F^{(-1)}_{1_\Gamma} \\
1_{\mathrm{G}} & \xrightarrow{F^{(0)}} & F_{1_\Gamma}
\end{array}
\qquad
\begin{array}{ccc}
\overline{F_\gamma} \otimes F_\gamma & \xrightarrow{F_\gamma^{(-1)} \otimes \mathrm{id}_{F_\gamma}} & F_{\gamma^{-1}} \otimes F_\gamma \\
\mathrm{ev}_{F_\gamma}\big\downarrow & & \big\downarrow F^{(2)}_{\gamma^{-1},\gamma} \\
1_{\mathrm{G}} & \xrightarrow[F^{(0)}]{} & F_{1_\Gamma}
\end{array}
\qquad
\begin{array}{ccc}
\overline{\overline{F_\gamma}} & \xrightarrow{\overline{F_\gamma^{(-1)}}} & \overline{F_{\gamma^{-1}}} \\
& {\scriptstyle \mathrm{bb}_{F_\gamma}}\nwarrow \quad \swarrow {\scriptstyle F^{(-1)}_{\gamma^{-1}}} & \\
& F_\gamma &
\end{array}
$$

$$
\begin{array}{ccccc}
\overline{F_{\gamma_1} \otimes F_{\gamma_2}} & \xrightarrow{\Upsilon_{F_{\gamma_1}, F_{\gamma_2}}} & \overline{F_{\gamma_2}} \otimes \overline{F_{\gamma_1}} & \xrightarrow{F^{(-1)}_{\gamma_2} \otimes F^{(-1)}_{\gamma_1}} & F_{\gamma_2^{-1}} \otimes F_{\gamma_1^{-1}} \\
\overline{F^{(2)}_{\gamma_1,\gamma_2}}\big\downarrow & & & & \big\downarrow F^{(2)}_{\gamma_2^{-1},\gamma_1^{-1}} \\
\overline{F_{\gamma_1\gamma_2}} & & \xrightarrow[F^{(-1)}_{\gamma_1\gamma_2}]{} & & F_{\gamma_2^{-1}\gamma_1^{-1}}
\end{array}
$$

# Example

Let $B$ be a unital pre-$C^*$-algebra.

We define an injective bar homomorphism $\tau : \mathrm{Aut}(B) \to \mathrm{Pic}(B)$ as follows:

1. given $\phi \in \mathrm{Aut}(B)$, let $\tau_\phi := \{b_\phi \mid b \in B\}$ be $B$ as a $\mathbf{R}$-vector space with
$$\forall a, b, c \in B, \qquad a \cdot b_\phi \cdot c := (ab\phi(c))_\phi,$$
$$\forall a, b \in B, \qquad (a_\phi, b_\phi) := \phi^{-1}(a^* b);$$
2. set $\tau^{(0)} := \mathrm{id}_B$;
3. given $\phi, \psi \in \mathrm{Aut}(B)$, define $\tau^{(2)}_{\phi,\psi} : \tau_\phi \otimes_B \tau_\psi \to \tau_{\phi\psi}$ by
$$\forall a, b \in B, \quad \tau^{(2)}_{\phi,\psi}(a_\phi \otimes b_\psi) := (a\phi(b))_{\phi\psi};$$
4. given $\phi \in \mathrm{Aut}(B)$, define $\tau^{(-1)}_\phi : \overline{\tau_\phi} \to \tau_{\phi^{-1}}$ by
$$\forall b \in B, \qquad \tau^{(-1)}_\phi(\overline{b_\phi}) := \phi^{-1}(b^*)_{\phi^{-1}}.$$

# Morphisms between bar homomorphisms

## Definition (cf. Baez–Lauda '04, Egger '11)

Let $\Gamma$ be a group and let $G$ be a coherent 2-group. We denote by $\mathrm{HOM}^{\mathrm{bar}}(\Gamma, G)$ the following groupoid:

1. an object is a bar homomorphism $F : \Gamma \longrightarrow G$;
2. an arrow $\phi : R \longrightarrow S$ is a family of arrows $\{\phi_\gamma : R_\gamma \longrightarrow S_\gamma\}_{\gamma\in\Gamma}$ making the following diagrams commute for all $\gamma, \gamma_1, \gamma_2 \in \Gamma$:

$$\begin{array}{ccc} & 1_G & \\ {\scriptstyle R^{(0)}}\swarrow & & \searrow{\scriptstyle S^{(0)}} \\ R_{1_\Gamma} & \xrightarrow[\phi_{1_\Gamma}]{} & S_{1_\Gamma} \end{array} \qquad \begin{array}{ccc} R_{\gamma_1} \otimes R_{\gamma_2} & \xrightarrow{\phi_{\gamma_1}\otimes\phi_{\gamma_2}} & S_{\gamma_1} \otimes S_{\gamma_2} \\ {\scriptstyle R^{(2)}_{\gamma_1,\gamma_2}}\downarrow & & \downarrow{\scriptstyle S^{(2)}_{\gamma_1,\gamma_2}} \\ R_{\gamma_1\gamma_2} & \xrightarrow[\phi_{\gamma_1\gamma_2}]{} & S_{\gamma_1\gamma_2} \end{array} \qquad \begin{array}{ccc} \overline{R_\gamma} & \xrightarrow{\overline{\phi_\gamma}} & \overline{S_\gamma} \\ {\scriptstyle R^{(-1)}_\gamma}\downarrow & & \downarrow{\scriptstyle S^{(-1)}_\gamma} \\ R_{\gamma^{-1}} & \xrightarrow[\phi_{\gamma^{-1}}]{} & S_{\gamma^{-1}} \end{array}$$

## Remark

If $\Gamma_1$ and $\Gamma_2$ are groups, then $\mathrm{HOM}^{\mathrm{bar}}(\Gamma_1, \Gamma_2) = \mathrm{HOM}(\Gamma_1, \Gamma_2) = \mathrm{Hom}(\Gamma_1, \Gamma_2)$.

# Abstract Cuntz–Pimsner for 2-groups

## Proposition (trivial exercise in undergraduate algebra)

*Let $G$ be a group. Then $(f \mapsto f(1)) : \mathrm{Hom}(\mathbf{Z}, G) \to G$ is a bijection.*

## Theorem (Ć., cf. Beggs–Brzeziński '14, Arici–Kaad–Landi '16)

*Let* $\mathsf{G}$ *be a 2-group. Define a functor* $\hat{\imath} : \textsc{Hom}^{\mathrm{bar}}(\mathbf{Z}, \mathsf{G}) \to \mathsf{G}$ *by*

$$\forall(\phi : R \to S) \in \mathsf{G}, \qquad \hat{\imath}(\phi) \coloneqq (\phi_1 : R_1 \to S_1).$$

*There exists a functor* $\mathcal{T} : \mathsf{G} \to \textsc{Hom}^{\mathrm{bar}}(\mathbf{Z}, \mathsf{G})$*, such that*

$$\hat{\imath} \circ \mathcal{T} = \mathrm{id}_{\mathsf{G}}, \quad \mathcal{T} \circ \hat{\imath} \simeq \mathrm{id}_{\textsc{Hom}^{\mathrm{bar}}(\mathbf{Z}, \mathsf{G})}.$$

In other words:

- $F : \mathbf{Z} \to \mathsf{G}$ is uniquely determined by $F_1$ up to natural isomorphism;
- $g \in \mathrm{Ob}(\mathsf{G})$ uniquely generates $\mathcal{T}(g) : \mathbf{Z} \to \mathsf{G}$ up to natural isomorphism.

# Principal U(1)-pre-C*-algebras

## Definition (cf. Arici–Kaad–Landi '16, Arici–D'Andrea–Landi '16)

Let $P$ be a unital pre-$C^*$-algebra with a $\mathrm{U}(1)$-action $\alpha : \mathrm{U}(1) \to \mathrm{Aut}(P)$ by isometric $*$-automorphisms; define

$$\forall k \in \mathbf{Z}, \qquad P_k \coloneqq \{p \in P \mid \forall z \in \mathrm{U}(1),\ \alpha_z(p) = z^k p\}.$$

We say that $P$ is a *principal* $\mathrm{U}(1)$*-pre-*$C^*$*-algebra* whenever $P = \oplus_{k \in \mathbf{Z}} P_k$ and

$$\exists e_1, \dots, e_m \in P_1 \qquad 1 = \sum_{i=1}^{m} e_i e_i^*,$$

$$\exists f_1, \dots, f_n \in P_1, \qquad 1 = \sum_{j=1}^{n} f_j^* f_j.$$

## Example

Given $q \in \mathbf{R} \setminus \{0, \pm 1\}$, the quantum group $\mathcal{O}_q(\mathrm{SU}_2)$ with $a, c \in \mathcal{O}_q(\mathrm{SU}_2)_1$ and

$$1 = aa^* + (qc)(qc)^*, \quad 1 = a^*a + c^*c.$$

# Associated line modules

Let $B$ be a unital pre-$C^*$-algebra.

We denote by $\textsc{Circ}(B)$ the following groupoid:

1. an object is a principal $\mathrm{U}(1)$-pre-$C^*$-algebra $P$ together with a $*$-isomorphism $\iota_P : B \longrightarrow P_0$;
2. an arrow $f : P \longrightarrow Q$ is a $\mathrm{U}(1)$-equivariant isometric $*$-isomorphism, such that $f \circ \iota_P = \iota_Q$.

## Definition

Given $P \in \mathrm{Ob}(\textsc{Circ}(B))$, define a bar homomorphism $\mathcal{L}(P) : \mathbf{Z} \longrightarrow \textsc{Pic}(B)$ thus:

1. given $k \in \mathbf{Z}$, let $\mathcal{L}(P)_k \coloneqq P_k$ as a $\mathbf{C}$-vector space with

$$\begin{aligned} \forall b_1, b_2 \in B,\ \forall p \in P_k, &\qquad b_1 \cdot p \cdot b_2 \coloneqq \iota_P(b_1)\, p\, \iota_P(b_2), \\ \forall p_1, p_2 \in P_k, &\qquad (p_1, p_2) \coloneqq \iota_P^{-1}(p_1^* p_2); \end{aligned}$$

2. set $\mathcal{L}(P)^{(0)} \coloneqq \iota_P$, induce $\mathcal{L}(P)^{(2)}_{\bullet,\bullet}$ from multiplication in $P$ and $\mathcal{L}(P)^{(-1)}_{\bullet}$ from the $*$-operation of $P$.

# Orbitwise Tannaka–Krein

**Theorem** (cf. Pimsner '97, Abadie–Eilers–Exel '98, Arici–Kaad–Landi '16)

*Define a functor* $\mathcal{L} : \textsc{Circ}(B) \to \textsc{Hom}^{\mathrm{bar}}(\mathbf{Z}, \textsc{Pic}(B))$ *by*

$$\forall(\phi : P \to Q) \in \textsc{Circ}(B),\ \forall k \in \mathbf{Z}, \quad \mathcal{L}(\phi)_k \coloneqq (\phi_k : \mathcal{L}(P)_k \to \mathcal{L}(Q)_k).$$

*There exists a functor* $\Sigma : \textsc{Hom}^{\mathrm{bar}}(\mathbf{Z}, \textsc{Pic}(B)) \to \textsc{Circ}(B)$, *such that*

$$\mathcal{L} \circ \Sigma = \mathrm{id}_{\textsc{Hom}^{\mathrm{bar}}(\mathbf{Z}, \textsc{Pic}(B))}, \quad \Sigma \circ \mathcal{L} \simeq \mathrm{id}_{\textsc{Circ}(B)}.$$

## Example

Given $\phi \in \mathrm{Aut}(B)$, we have $B \rtimes^{\mathrm{alg}}_{\phi} \mathbf{Z} \in \mathrm{Ob}(\textsc{Circ}(B))$ and

$$\mathcal{L}(B \rtimes^{\mathrm{alg}}_{\phi} \mathbf{Z}) = \tau \circ (k \mapsto \phi^k).$$

## Example

Given $q \in \mathbf{R} \setminus \{0, \pm 1\}$, we have $\mathcal{O}_q(\mathrm{SU}_2) \in \mathrm{Ob}(\textsc{Circ}(\mathcal{O}_q(\mathbf{CP}^1)))$ and that $\mathcal{L}(\mathcal{O}_q(\mathrm{SU}_2))$ recovers the relative line modules $\{\mathcal{E}_k\}_{k \in \mathbf{Z}}$ on $\mathcal{O}_q(\mathbf{CP}^1)$.

# Theme

Corollary (Pimsner '97, Abadie–Eilers–Exel '98...)

*Define a functor* $\mathcal{L}_1 : \textsc{Circ}(B) \to \textsc{Pic}(B)$ *by* $\mathcal{L}_1 := \hat{\imath} \circ \mathcal{L}$, *e.g.,*

$$\forall P \in \mathrm{Ob}(\textsc{Circ}(B)),\ \forall k \in \mathbf{Z}, \qquad \mathcal{L}_1(P) := \mathcal{L}(P)_1.$$

*There exists a functor* $\mathcal{P} : \textsc{Pic}(B) \to \textsc{Circ}(B)$, *e.g.,* $\mathcal{P} := \Sigma \circ \mathcal{T}$, *such that*

$$\mathcal{L}_1 \circ \mathcal{P} = \mathrm{id}_{\textsc{Pic}(B)}, \quad \mathcal{P} \circ \mathcal{L}_1 \simeq \mathrm{id}_{\textsc{Circ}(B)}.$$

Such $\mathcal{P} : \textsc{Pic}(B) \to \textsc{Circ}(B)$ is Cuntz–Pimsner for Morita auto-equivalences.

## Example

Given $\phi \in \mathrm{Aut}(B)$, we have $\mathcal{L}_1(B \rtimes^{\mathrm{alg}}_{\phi} \mathbf{Z}) = \tau_\phi$.

## Example

Given $q \in \mathbf{R} \setminus \{0, \pm 1\}$, we have $\mathcal{L}_1(\mathcal{O}_q(\mathrm{SU}_2)) = \mathcal{O}_q(\mathrm{SU}_2)_1 = \mathcal{E}_1$, the relative line module on $\mathcal{O}_q(\mathbf{CP}^1)$ generated by $\{a, c\} \subset \mathcal{O}_q(\mathrm{SU}_2)_1$.

# Hermitian right connections

Fix a unital pre-$C^*$-algebra $B$ with $*$-differential calculus $\Omega_B$.

## Definition

A *Hermitian right connection* on a Hermitian line $B$-module $E$ is a $\mathbf{C}$-linear map $\nabla : E \to E \otimes \Omega^1_B$ satisfying:

$$\forall e \in E, \forall b \in B, \quad \nabla(e \cdot b) = \nabla(e) \cdot b + e \otimes \mathrm{d}_B b,$$
$$\forall e_1, e_2 \in E, \quad \mathrm{d}(e_1, e_2) = \nabla(e_1)^*_{\langle 1 \rangle} \cdot (\nabla(e_1)_{\langle 0 \rangle}, e_2) + (e_1, \nabla(e_2)_{\langle 0 \rangle}) \cdot \nabla(e_2)_{\langle 1 \rangle},$$

where we apply to $\eta \in E \otimes \Omega_B$ the Sweedleresque notation $\eta = \eta_{\langle 0 \rangle} \otimes \eta_{\langle 1 \rangle}$

## Proposition

*A Hermitian right connection $\nabla$ on a Hermitian line $B$-module $E$ uniquely extends to a $\mathbf{C}$-linear map $\nabla : E \otimes \Omega^\bullet_B \to E \otimes \Omega^{\bullet+1}_B$, such that*

$$\forall \eta \in E, \forall \beta \in \Omega_B, \qquad \nabla(\eta \cdot \beta) = \nabla(\eta) \cdot \beta + (-1)^{|\eta|} \eta \cdot \mathrm{d}_B \beta.$$

# Hermitian line connections

## Definition (Beggs–Majid '18)

A *Hermitian line connection* on a Hermitian line $B$-module $E$ is $(\sigma_E, \nabla_E)$, where:

1. $\sigma_E : \Omega^\bullet_B \otimes_B E \to E \otimes_B \Omega^\bullet_B$ is a $B$-bimodule isomorphism restricting to elementary $B \otimes_B E \to E \otimes_B B$, making $E \otimes_B \Omega_B$ a graded $\Omega_B$-bimodule via

$$\forall \alpha, \beta \in \Omega_B, \forall \eta \in E, \qquad \alpha \cdot \eta \cdot \beta := \sigma_E(\alpha \otimes \eta_{\langle 0 \rangle}) \otimes \eta_{\langle 1 \rangle} \cdot \beta;$$

2. $\nabla_E$ is a Hermitian right connection on $E$ satisfying

$$\forall \beta \in \Omega_B, \forall \eta \in E \otimes_B \Omega_B, \quad \nabla_E(\beta \cdot \eta) = d_B \beta \cdot \eta + (-1)^{|\beta|} \beta \cdot \nabla_E(\eta).$$

Its *conjugate* is the Hermitian line connection $(\sigma_{\overline{E}}, \nabla_{\overline{E}})$ on $\overline{E}$ defined by

$$\begin{aligned} \forall e \in E, \forall \beta \in \Omega_B, \quad \sigma_{\overline{E}}(\beta \otimes \overline{e}) &:= \overline{\sigma_E^{-1}(e \otimes \beta^*)_{\langle 0 \rangle}} \otimes \sigma_E^{-1}(e \otimes \beta^*)^*_{\langle -1 \rangle}, \\ \forall e \in E, \qquad \nabla_{\overline{E}}(\overline{e}) &:= -\overline{(\sigma_E^{-1} \circ \nabla_E(e))_{\langle 0 \rangle}} \otimes (\sigma_E^{-1} \circ \nabla_E(e))^*_{\langle -1 \rangle}, \end{aligned}$$

where we apply to $\eta \in \Omega_B \otimes E$ the Sweedleresque notation $\eta = \eta_{\langle -1 \rangle} \otimes \eta_{\langle 0 \rangle}$.

# The differentiable Picard 2-group

## Definition (Beggs–Majid '18 *mutatis mutandis*)

The *differentiable Picard 2-group* of $B$ with respect to $\Omega_B$ is the coherent 2-group $\widehat{\mathrm{Pic}}(B)$ defined as follows:

1. an object is a pair $(E, \nabla_E)$, where $E \in \mathrm{Ob}(\mathrm{Pic}(B))$ and $(\sigma_E, \nabla_E)$ is a Hermitian line connection on $E$.
2. an arrow $\phi : (E, \nabla_E) \to (F, \nabla_F)$ is $\phi : E \to F$ in $\mathrm{Pic}(B)$, such that
$$\nabla_F \circ \phi = (\phi \otimes \mathrm{id}) \circ \nabla_E.$$
3. we set $(E, \nabla_E) \otimes (F, \nabla_F) := (E \otimes_B F, \nabla_{E \otimes_B F})$, where
$$\sigma_{E\otimes_B F} := (\mathrm{id} \otimes \sigma_F) \circ (\sigma_E \otimes \mathrm{id}),$$
$$\nabla_{E\otimes_B F} := (\mathrm{id} \otimes \sigma_F) \circ (\nabla_E \otimes \mathrm{id}) + \mathrm{id} \otimes \nabla_F;$$
4. we set $\overline{(E, \nabla_E)} := (\overline{E}, \nabla_{\overline{E}})$, where $(\sigma_{\overline{E}}, \nabla_{\overline{E}})$ is conjugate of $(\sigma_E, \nabla_E)$.

Here, composition is composition in $\mathrm{Pic}(B)$, the unit is $(B, \mathrm{d}_B, \mathrm{id}_{\Omega_B})$ and the associator, unit, left unitor, right unitor, and evaluation are those of $\mathrm{Pic}(B)$.

# Example

From now on, let $\widehat{\mathrm{Aut}}(B) \coloneqq \mathrm{Aut}(\Omega_B, \mathrm{d})$.

We can refine $\tau \in \mathrm{HOM}^{\mathrm{bar}}(\mathrm{Aut}(B), \mathrm{PIC}(B))$ to $\hat{\tau} \in \mathrm{HOM}^{\mathrm{bar}}(\widehat{\mathrm{Aut}}(B), \widehat{\mathrm{PIC}}(B))$ thus:

1. given $\phi \in \widehat{\mathrm{Aut}}(B)$, let $\hat{\tau}_\phi \coloneqq (\tau_\phi, \mathrm{d}_\phi)$, where
$$\forall a, b \in B, \qquad \sigma_\phi(\mathrm{d}_B a \otimes b) \coloneqq 1_\phi \otimes (\mathrm{d}_B \circ \phi^{-1})(a) \cdot \phi^{-1}(b),$$
$$\forall b \in B, \qquad \mathrm{d}_\phi(b_\phi) \coloneqq 1_\phi \otimes (\mathrm{d}_B \circ \phi^{-1})(b);$$
2. set $\hat{\tau}^{(0)} \coloneqq \tau^{(0)} \coloneqq \mathrm{id}_B$;
3. given $\phi, \psi \in \widehat{\mathrm{Aut}}(B)$, define $\hat{\tau}^{(2)}_{\phi,\psi} : \hat{\tau}_\phi \otimes \hat{\tau}_\psi \to \hat{\tau}_{\phi\psi}$ by
$$\forall a, b \in B, \quad \hat{\tau}^{(2)}_{\phi,\psi}(a_\phi \otimes b_\psi) \coloneqq \tau^{(2)}_{\phi,\psi} \coloneqq (a\phi(b))_{\phi\psi};$$
4. given $\phi \in \widehat{\mathrm{Aut}}(B)$, define $\hat{\tau}^{(-1)}_\phi : \overline{\tau_\phi} \to \tau_{\phi^{-1}}$ by
$$\forall b \in B, \qquad \hat{\tau}^{(-1)}_\phi(\overline{b_\phi}) \coloneqq \tau^{(-1)}_\phi(\overline{b_\phi}) \coloneqq \phi^{-1}(b^*)_{\phi^{-1}}.$$

# Horizontal calculi

## Definition (Đurđević '98, Ć. '21)

Let $P \in \mathrm{Ob}(\textsc{Circ}(B))$. A *horizontal calculus* for $P$ consists of:

1. a $\mathrm{U}(1)$-equivariant *curved* $*$-differential calculus $\Omega_{P,\mathrm{h}}$ over $P$ (i.e., $\mathrm{d}_{P,\mathrm{h}}^2 \neq 0$),
2. an extension $\hat{\iota}_P : \Omega_B^\bullet \to (\Omega_{P,\mathrm{h}}^\bullet)^{\mathrm{U}(1)}$ of $\iota_P : B \to P^{\mathrm{U}(1)}$ to an isomorphism of graded $*$-algebras,

such that

$$\left((\Omega_{P,\mathrm{h}}^\bullet)^{\mathrm{U}(1)}, \mathrm{d}_{P,\mathrm{h}}\restriction_{(\Omega_{P,\mathrm{h}}^\bullet)^{\mathrm{U}(1)}}\right) = (\hat{\iota}_P(\Omega_B^\bullet), \hat{\iota}_P \circ \mathrm{d}_B).$$

We define a groupoid $\widehat{\textsc{Circ}}(B)$ as follows:

1. an object is $(P, \Omega_{P,\mathrm{h}})$, where $P \in \mathrm{Ob}(\textsc{Circ}(B))$ and $\Omega_{P,\mathrm{h}}$ is a horizontal calculus on $P$;
2. an arrow $f : (P, \Omega_{P,\mathrm{h}}) \to (Q, \Omega_{Q,\mathrm{h}})$ is an arrow $f : P \to Q$ in $\textsc{Circ}(B)$ extending to a $\mathrm{U}(1)$-equivariant isomorphism $f : \Omega_{P,\mathrm{h}} \to \Omega_{Q,\mathrm{h}}$ of graded $*$-algebras, such that

$$f \circ \mathrm{d}_{P,\mathrm{h}} = \mathrm{d}_{Q,\mathrm{h}} \circ f.$$

# Associated Hermitian line connections

## Definition (Ć., cf. Ć.–Mesland '21)

Given $(P, \Omega_{P,\mathrm{h}}) \in \mathrm{Ob}(\widehat{\mathrm{CIRC}}(B))$, refine $\mathcal{L}(P) : \mathbf{Z} \to \mathrm{PIC}(B)$ to a bar homomorphism $\hat{\mathcal{L}}(P, \Omega_{P,\mathrm{h}}) : \mathbf{Z} \to \widehat{\mathrm{PIC}}(B)$ thus:

1. given $k \in \mathbf{Z}$, let $\hat{\mathcal{L}}(P, \Omega_{P,\mathrm{h}})_k \coloneqq (\mathcal{L}(P)_k, \nabla_{P;k})$, where

$$\forall \beta \in \Omega_B,\ \forall p \in P_k, \quad \sigma_{P;k}(\hat{\iota}_P(\beta) \otimes p) \coloneqq \sum_{i=1}^{m} e_i \otimes \hat{\iota}_P^{-1}(e_i^* \cdot \hat{\iota}_P(\beta) \cdot p),$$

$$\forall p \in P_k, \qquad \nabla_{P;k}(p) \coloneqq \sum_{i=1}^{m} e_i \otimes \hat{\iota}_P^{-1}(e_i^* \cdot \mathrm{d}_{P,\mathrm{h}}(p)),$$

   for any $\{e_1, \dots, e_m\} \subset P_k$ such that $\sum_{i=1}^{m} e_i e_i^* = 1$;

2. set $\hat{\mathcal{L}}(P, \Omega_{P,\mathrm{h}})^{(0)} \coloneqq \iota_P$, induce $\hat{\mathcal{L}}(P, \Omega_{P,\mathrm{h}})^{(2)}_{\bullet,\bullet}$ from multiplication in $\Omega_{P,\mathrm{h}}$ and $\hat{\mathcal{L}}(P, \Omega_{P,\mathrm{h}})^{(-1)}_{\bullet}$ from the $*$-operation of $\Omega_{P,\mathrm{h}}$.

# A differentiable orbitwise Tannaka–Krein theorem

## Theorem (Ć.)

*Define a functor* $\hat{\mathcal{L}} : \widehat{\text{CIRC}}(B) \longrightarrow \text{HOM}^{\text{bar}}(\mathbf{Z}, \widehat{\text{PIC}}(B))$ *by*

$$\forall(\phi : (P, \Omega_{P,\mathrm{h}}) \longrightarrow (Q, \Omega_{P,\mathrm{h}})) \in \widehat{\text{CIRC}}(B),\ \forall k \in \mathbf{Z},$$
$$\hat{\mathcal{L}}(\phi)_k := (\phi_k : \hat{\mathcal{L}}(P, \Omega_{P,\mathrm{h}})_k \longrightarrow \hat{\mathcal{L}}(Q, \Omega_{Q,\mathrm{h}})_k).$$

*There exists a functor* $\hat{\Sigma} : \text{HOM}^{\text{bar}}(\mathbf{Z}, \widehat{\text{PIC}}(B)) \longrightarrow \widehat{\text{CIRC}}(B)$, *such that*

$$\hat{\mathcal{L}} \circ \hat{\Sigma} = \mathrm{id}_{\text{HOM}^{\text{bar}}(\mathbf{Z}, \widehat{\text{PIC}}(B))}, \quad \hat{\Sigma} \circ \hat{\mathcal{L}} \simeq \mathrm{id}_{\widehat{\text{CIRC}}(B)}.$$

## Example

Let $\phi \in \widehat{\mathrm{Aut}}(B)$. Then

$$\hat{\Sigma}(\tau \circ (k \mapsto \phi^k)) \cong \left(B \rtimes_{\phi}^{\mathrm{alg}} \mathbf{Z}, (\Omega_B \rtimes_{\phi_*}^{\mathrm{alg}} \mathbf{Z}, \mathrm{id}_{\mathbf{C}[\mathbf{Z}]} \otimes \mathrm{d})\right)$$

# A variation on a theme

## Corollary

*Define a functor* $\hat{\mathcal{L}}_1 : \widehat{\mathrm{CIRC}}(B) \to \widehat{\mathrm{PIC}}(B)$ *by* $\hat{\mathcal{L}}_1 \coloneqq \hat{\iota} \circ \hat{\mathcal{L}}$*, e.g.,*

$$\forall (P, \Omega_{P,\mathrm{h}}) \in \mathrm{Ob}(\widehat{\mathrm{CIRC}}(B)), \qquad \hat{\mathcal{L}}_1(P, \Omega_{P,\mathrm{h}}) \coloneqq (\mathcal{L}(P)_1, \nabla_{P;1}).$$

*There exists a functor* $\hat{\mathcal{P}} : \widehat{\mathrm{PIC}}(B) \to \widehat{\mathrm{CIRC}}(B)$*, e.g.,* $\hat{\mathcal{P}} \coloneqq \hat{\Sigma} \circ \mathcal{T}$*, such that*

$$\hat{\mathcal{L}}_1 \circ \hat{\mathcal{P}} = \mathrm{id}_{\widehat{\mathrm{PIC}}(B)}, \quad \hat{\mathcal{P}} \circ \hat{\mathcal{L}}_1 \simeq \mathrm{id}_{\widehat{\mathrm{CIRC}}(B)}.$$

Such $\hat{\mathcal{P}} : \widehat{\mathrm{PIC}}(B) \to \widehat{\mathrm{CIRC}}(B)$ is differentiable Cuntz–Pimsner for Hermitian line modules with Hermitian line connection!

## Example

Let $\phi \in \widehat{\mathrm{Aut}}(B)$, so that $(B \rtimes^{\mathrm{alg}}_{\phi} \mathbf{Z}, \Omega_B \rtimes^{\mathrm{alg}}_{\phi_*} \mathbf{Z}, \mathrm{id}_{\mathbf{C}[\mathbf{Z}]} \otimes \mathrm{d}) \in \mathrm{Ob}(\widehat{\mathrm{CIRC}}(B))$. Then

$$\mathcal{P}(\hat{\tau}_\phi) \cong \hat{\Sigma}(\hat{\tau} \circ (k \mapsto \phi^k)) \cong (B \rtimes^{\mathrm{alg}}_{\phi} \mathbf{Z}, (\Omega_B \rtimes^{\mathrm{alg}}_{\phi_*} \mathbf{Z}, \mathrm{id}_{\mathbf{C}[\mathbf{Z}]} \otimes \mathrm{d})).$$

# Curvature

Let $\widehat{\mathrm{Pic}}(B)$ the group of isomorphism classes in $\widehat{\mathrm{PIC}}(B)$ with respect to $\otimes$.

## Definition (Beggs–Majid '18)

Let $(E, \nabla_E) \in \mathrm{Ob}(\widehat{\mathrm{PIC}}(B))$. The *curvature* of $[(E, \nabla_E)] \in \widehat{\mathrm{Pic}}(B)$ is the unique 2-form $\mathbf{F}_{[(E,\nabla_E)]} \in \mathrm{Z}_B(\Omega^2_B)$ satisfying

$$\forall e \in E, \qquad -\mathrm{i}\nabla_E^2(e) = e \otimes \mathbf{F}_{[(E,\nabla_E)]}.$$

## Example

For every $\phi \in \widehat{\mathrm{Aut}}(B)$, we have $\mathbf{F}_{[\hat{\tau}_\phi]} = 0$.

## Proposition

*Let* $(P, \nabla_P) \in \mathrm{Ob}(\widehat{\mathrm{CIRC}}(B))$. *Then*

$$\forall k \in \mathbf{Z},\ \forall p \in P_k, \qquad -\mathrm{i}\,\mathrm{d}^2_{P,\mathrm{h}}(p) = p \cdot \hat{\iota}_P(\mathbf{F}_{[(\hat{\mathcal{L}}(P)_k, \nabla_{P;k})]}).$$

# Curvature as a 1-cocycle

## Proposition (Fröhlich '73, Beggs–Majid '18)

*There is a unique homomorphism* $\hat{\hat{\Phi}} : \widehat{\mathrm{Pic}}(B) \to \mathrm{Aut}(\mathrm{Z}_B(\Omega_B))$, *such that*

$$\forall (E, \nabla_E) \in \mathrm{Ob}(\widehat{\mathrm{Pic}}(B)),\ \forall e \in E,\ \forall \beta \in \mathrm{Z}_B(\Omega_B), \quad e \otimes \hat{\hat{\Phi}}_{[(E,\nabla_E)]}(\beta) = \sigma_E(\beta \otimes e).$$

## Proposition (Beggs–Majid '18)

*The assignment* $[(E, \nabla_E)] \mapsto \mathbf{F}_{[(E,\nabla_E)]}$ *defines a 1-cocycle* $\mathbf{F} : \widehat{\mathrm{Pic}}(B) \to (\mathrm{Z}_B(\Omega_B^2), +)$ *with respect to* $\hat{\hat{\Phi}}$, *i.e.,*

$$\forall (E, \nabla_E), (F, \nabla_F) \in \mathrm{Ob}(\widehat{\mathrm{Pic}}(B)),$$
$$\mathbf{F}_{[(E \otimes_B F, \nabla_{E \otimes_B F})]} = \mathbf{F}_{[(E,\nabla_E)]} + \hat{\hat{\Phi}}_{[(E,\nabla_E)]}(\mathbf{F}_{[(F,\nabla_F)]}).$$

# Preliminaries

Let $\kappa \in \mathbf{R} \setminus \{0\}$ be given.

For $n \in \mathbf{Z}$, let $[n]_\kappa := n$ if $\kappa = 1$, otherwise $[n]_\kappa := \frac{\kappa^n - 1}{\kappa - 1}$.

Let $\Lambda_\kappa := \mathbf{C}[\mathrm{d}_\kappa t]/\langle(\mathrm{d}_\kappa t)^2\rangle$ with $\mathrm{d}_\kappa t$ skew-adjoint.

Given a graded $\mathrm{U}(1)$-$*$-algebra $\Omega$, let

$$\Lambda_\kappa \rtimes \Omega := \Lambda_\kappa \mathbin{\widehat{\otimes}} \Omega / \langle \mathrm{d}_\kappa t \mathbin{\widehat{\otimes}} \omega - (-1)^{|\omega|} \alpha_{\mathrm{i} \log \kappa}(\omega) \mathbin{\widehat{\otimes}} \mathrm{d}_\kappa t \mid \omega \in \Omega \rangle$$

with $\mathrm{d}_\kappa t$ defined to be $\mathrm{U}(1)$-invariant.

## Definition (cf. Đurđević ’97)

The $\kappa$-*vertical calculus* of $P \in \mathrm{Ob}(\textsc{Circ}(B))$ is the $\mathrm{U}(1)$-equivariant $*$-differential calculus $\Omega_{P,\mathrm{v};\kappa} := \Lambda_\kappa \rtimes P$ on $P$ with

$$\forall n \in \mathbf{Z},\ \forall p \in P_n, \qquad \mathrm{d}_{P,\mathrm{v};\kappa}(p) := \mathrm{d}_\kappa t \cdot 2\pi\mathrm{i}[n]_\kappa p.$$

# Quantum principal $\mathrm{U}_\kappa(1)$-bundles

Definition (Brzeziński–Majid '93, Đurđević '97, Beggs–Majid '21)

A *quantum principal* $\mathrm{U}_\kappa(1)$*-bundle* over $B$ is a pair $(P, \Omega_P)$, where:

1. $P \in \mathrm{Ob}(\textsc{Circ}(B))$ with given $*$-isomorphism $\iota_P : B \to P^{\mathrm{U}(1)}$,
2. $\Omega_P$ is a $\mathrm{U}(1)$-equivariant $*$-differential calculus on $P$ with an injective extension of $\iota_P$ to a graded $*$-homomorphism $\hat{\iota}_P : \Omega_B^\bullet \to (\Omega_P^\bullet)^{\mathrm{U}(1)}$

such that

1. $\mathrm{d}_P \circ \hat{\iota}_P = \hat{\iota}_P \circ \mathrm{d}_B$;
2. we can define surjective $\mathrm{ver}_{P;\kappa} : \Omega_P \to \Omega_{P,\mathrm{v};\kappa}$ with kernel $\Omega_P \cdot \Omega_B^1$ by

$$\mathrm{ver}_{P;\kappa}\restriction_P = \mathrm{id}_P, \quad \mathrm{ver}_{P;\kappa} \circ \mathrm{d}_P\restriction_P = \mathrm{d}_{P,\mathrm{v};\kappa},$$

3. we can define $\mathrm{int}_{P;\kappa} : \Omega_p \to \Lambda_\kappa \rtimes \Omega_P$ with $\ker(\mathrm{id} - \mathrm{int}_{P;\kappa}) = P \cdot \Omega_B$ by

$$\mathrm{int}_{P;\kappa}\restriction_P = \mathrm{id}_P, \quad \mathrm{int}_{P;\kappa}\restriction_{\Omega_P^1} = \mathrm{id}_{\Omega_P^1} + \mathrm{ver}_{P;\kappa}\restriction_{\Omega_P^1}.$$

# Strong connections

## Definition (Brzeziński–Majid '93, Hajac '96, Đurđević '97)

A *strong connection* on a quantum principal $\mathrm{U}(1)$-bundle $(P, \Omega_P)$ over $(B, \Omega_B)$ is a $\mathrm{U}(1)$-equivariant morphism of $P$-$*$-bimodules $\Pi : \Omega^1_P \to \Omega^1_P$, such that:

1. $\mathrm{id}_P$ extends via $\Pi$ to $\mathrm{ver}_\Pi : \Omega_P \to \Omega_P$, such that

$$(\mathrm{ver}_\Pi)^2 = \mathrm{ver}_\Pi, \quad \ker \mathrm{ver}_\Pi = \ker \mathrm{ver}_{P;\kappa};$$

2. $\mathrm{id}_P$ extends via $\mathrm{id}_{\Omega^1_P} - \Pi$ to $\mathrm{hor}_\Pi : \Omega_P \to \Omega_P$, such

$$(\mathrm{hor}_\Pi)^2 = \mathrm{hor}_\Pi, \quad \operatorname{ran} \mathrm{hor}_\Pi = \ker \mathrm{int}_{P;\kappa}\,.$$

## Example (Brzeziński–Majid '93)

Let $q \in \mathbf{R} \setminus \{0, \pm 1\}$, let $\Omega_{q,3}(\mathrm{SU}_2)$ be the 3-dimensional calculus on $\mathcal{O}_q(\mathrm{SU}_2)$. The *q-monopole* is a strong connection $\Pi_q$ on the quantum principal $\mathrm{U}_{q^2}(1)$-bundle $(\mathcal{O}_q(\mathrm{SU}_2), \Omega_{q,3}(\mathrm{SU}_2))$ over $(\mathcal{O}_q(\mathbf{CP}^1), \Omega_q(\mathbf{CP}^1))$.

# Horizontal calculi from strong connections

Define a groupoid $\textsc{Gauge}_\kappa(B)$ as follows:

1. an object is a triple $(P, \Omega_P, \Pi)$, where $\Pi$ is a strong connection on the quantum principal $\mathrm{U}_\kappa(1))$-bundle $(P, \Omega_P)$;
2. an arrow $f : (P, \Omega_P, \Pi_P) \to (Q, \Omega_Q, \Pi_Q)$ is a $\mathrm{U}(1)$-equivariant isomorphism $f : \Omega_P^\bullet \to \Omega_Q^\bullet$ of graded $*$-algebras, such that

$$f \circ \hat{\iota}_P = \hat{\iota}_Q, \quad f \circ \mathrm{d}_P = \mathrm{d}_Q \circ f, \quad (\mathrm{id}_{\Lambda^1_\kappa} \otimes f) \circ \mathrm{int}_{P;\kappa} = \mathrm{int}_{P;\kappa} \circ f, \quad f \circ \Pi_P = \Pi_Q \circ f.$$

Define a functor $\mathrm{Hor}_\kappa : \textsc{Gauge}_\kappa(B) \to \widehat{\textsc{Circ}}(B)$ as follows:

$$\forall (f : (P, \Omega_P, \Pi_P) \to (Q, \Omega_Q, \Pi_Q)) \in \textsc{Gauge}_\kappa(B),$$
$$\mathrm{Hor}_\kappa(f) :=$$
$$(f : (P, (P \cdot \hat{\iota}_P(\Omega_B), (\mathrm{id} - \Pi_P) \circ \mathrm{d}_P)) \to (Q, (Q \cdot \hat{\iota}_Q(\Omega_B), (\mathrm{id} - \Pi_Q) \circ \mathrm{d}_Q)))$$

## Goal

Use $\mathrm{Hor}_\kappa$ to get a differentiable Cuntz–Pimsner construction for $\textsc{Gauge}_\kappa(B)$.

# Synthesis of total calculi

Say that $(P, \Omega_{P,\mathrm{h}}) \in \mathrm{Ob}(\widehat{\textsc{Circ}}(B))$ is $\kappa$-*adapted* if there exists a $\mathrm{U}(1)$-equivariant morphism of $P$-$*$-bimodules $R[\Omega_{P,\mathrm{h}}] : \Omega^1_{P,\mathrm{v};\kappa} \to \Omega^2_{P,\mathrm{v};\kappa}$, such that

$$\forall p \in P, \qquad \mathrm{d}^2_{P,\mathrm{h}}(p) = \mathrm{i}R[\Omega_{P,\mathrm{h}}](\mathrm{d}_{P,\mathrm{v};\kappa}(p)).$$

Let $\widehat{\textsc{Circ}}_\kappa(B)$ be the strictly full subgroupoid of $\widehat{\textsc{Circ}}(B)$ of all $\kappa$-adapted objects.

## Proposition (Đurđević '10, Ć. '21)

1. *The range of* $\mathrm{Hor}_\kappa : \textsc{Gauge}_\kappa(B) \to \widehat{\textsc{Circ}}(B)$ *is* $\widehat{\textsc{Circ}}_\kappa(B)$.
2. *There exists a functor* $\mathcal{S}_\kappa : \widehat{\textsc{Circ}}_\kappa(B) \to \widehat{\textsc{Gauge}}_\kappa(B)$, *such that*

$$\mathrm{Hor}_\kappa \circ \mathcal{S}_\kappa = \mathrm{id}_{\widehat{\textsc{Circ}}_\kappa(B)}, \quad \mathcal{S}_\kappa \circ \mathrm{Hor}_\kappa \simeq \mathrm{id}_{\widehat{\textsc{Circ}}_\kappa(B)}.$$

Given $(P, \Omega_{P,\mathrm{h}}) \in \mathrm{Ob}(\widehat{\textsc{Circ}}_\kappa(B))$, we have $\mathcal{S}_\kappa(P, \Omega_{P,\mathrm{h}}) \coloneqq (P, \Omega_P, \Pi_P)$, where

$$\Omega_P \coloneqq \Lambda_k \rtimes \Omega_{P,\mathrm{h}},$$
$$\mathrm{d}_P\restriction_P \coloneqq \mathrm{d}_{P,\mathrm{v};\kappa}\restriction_P + \mathrm{d}_{P,\mathrm{h}}\restriction_P, \quad \mathrm{d}_P(\mathrm{d}_\kappa t) \coloneqq -\mathrm{i}R[\Omega_{P,\mathrm{h}}](\mathrm{d}_\kappa t),$$
$$\Pi_P \coloneqq \mathrm{Proj}_{\Omega^1_{P,\mathrm{v};\kappa}}.$$

# Characterization of κ-adapted horizontal calculi

## Proposition

1. *Let* $\mathrm{HOM}^{\mathrm{bar}}_{\kappa}(\mathbf{Z}, \widehat{\mathrm{PIC}}(B))$ *be the strictly full subgroupoid of* $\mathrm{HOM}^{\mathrm{bar}}(\mathbf{Z}, \widehat{\mathrm{PIC}}(B))$ *of all S that satisfy*

$$\forall m, n \in \mathbf{Z}, \qquad \mathbf{F}_{[S_{m+n}]} = \mathbf{F}_{[S_m]} + \kappa^{-m}\mathbf{F}_{[S_n]}.$$

*Then* $\hat{\mathcal{L}} : \widehat{\mathrm{CIRC}}(B) \to \mathrm{HOM}^{\mathrm{bar}}(\mathbf{Z}, \widehat{\mathrm{PIC}}(B))$ *satisfies*

$$\hat{\mathcal{L}}(\widehat{\mathrm{CIRC}}_{\kappa}(B)) = \mathrm{HOM}^{\mathrm{bar}}_{\kappa}(\mathbf{Z}, \widehat{\mathrm{PIC}}(B)).$$

2. *Let* $\widehat{\mathrm{PIC}}_{\kappa}(B)$ *be the strictly full subgroupoid of* $\widehat{\mathrm{PIC}}(B)$ *of all* $(E, \nabla_E)$ *that satisfy*

$$\hat{\Phi}_{[(E,\nabla_E)]}(\mathbf{F}_{[(E,\nabla_E)]}) = \kappa^{-1}\mathbf{F}_{[(E,\nabla_E)]}.$$

*Then* $\hat{\imath} : \mathrm{HOM}^{\mathrm{bar}}(\mathbf{Z}, \widehat{\mathrm{PIC}}(B)) \to \widehat{\mathrm{PIC}}(B)$ *satisfies*

$$\hat{\imath}(\mathrm{HOM}^{\mathrm{bar}}_{\kappa}(\mathbf{Z}, \widehat{\mathrm{PIC}}(B))) = \widehat{\mathrm{PIC}}_{\kappa}(B).$$

# Another variation

Theorem (Ć.)

*The functor* $\hat{\mathcal{L}}_1 \circ \mathrm{Hor}_\kappa : \textsc{Gauge}_\kappa(B) \to \widehat{\textsc{Pic}}(B)$ *satisfies*

$$\hat{\mathcal{L}}_1 \circ \mathrm{Hor}_\kappa(\textsc{Gauge}_\kappa(B)) = \widehat{\textsc{Pic}}_\kappa(B).$$

*Hence, there exists a functor* $\mathcal{P}_\kappa : \widehat{\textsc{Pic}}_\kappa(B) \to \textsc{Gauge}_\kappa(B)$, *such that*

$$(\hat{\mathcal{L}}_1 \circ \mathrm{Hor}_\kappa) \circ \mathcal{P}_\kappa = \mathrm{id}_{\widehat{\textsc{Pic}}_\kappa(B)}, \quad \mathcal{P}_\kappa \circ (\hat{\mathcal{L}}_1 \circ \mathrm{Hor}_\kappa) \simeq \mathrm{id}_{\textsc{Gauge}_\kappa(B)}.$$

Such $\mathcal{P}_\kappa : \widehat{\textsc{Pic}}_\kappa(B) \to \textsc{Gauge}_\kappa(B)$ is differentiable Cuntz–Pimsner for quantum principal $\mathrm{U}_\kappa(1)$-bundles over $(B, \Omega_B)$ with strong (bimodule) connection!

# The $q$-monopole revisited

Let $q \in \mathbf{R} \setminus \{0, \pm 1\}$ be given. Let $(\mathcal{O}_q(\mathrm{SU}_2), \Omega_{q,3}(\mathrm{SU}_2), \Pi_q) \in \mathrm{Ob}(\textsc{Gauge}_{q^2}(B))$ be the $q$-monopole over $(\mathcal{O}_q(\mathbf{CP}^1), \Omega_q(\mathbf{CP}^1))$.

1. The bar homomorphism

$$\hat{\mathcal{L}} \circ \mathrm{Hor}(\mathcal{O}_q(\mathrm{SU}_2), \Omega_{q,3}(\mathrm{SU}_2), \Pi_q) \in \mathrm{Ob}(\textsc{Hom}^{\mathrm{bar}}_{q^2}(\mathbf{Z}, \widehat{\textsc{Pic}}(\mathcal{O}_q(\mathbf{CP}^1))))$$

   recovers $\{(\mathcal{E}_k, \nabla_k)\}_{k \in \mathbf{Z}}$, where $\nabla_k$ is the unique left $\mathcal{O}_q(\mathrm{SU}_2)$-covariant Hermitian right connection on the relative line module $\mathcal{E}_k$ over $\mathcal{O}_q(\mathbf{CP}^1)$.

2. Hence, in particular,

$$\forall m, n \in \mathbf{Z}, \qquad \mathbf{F}_{[(\mathcal{E}_{m+n}, \nabla_{m+n})]} = \mathbf{F}_{[(\mathcal{E}_m, \nabla_m)]} + q^{-2m} \mathbf{F}_{[(\mathcal{E}_n, \nabla_n)]}.$$

3. Hence, in particular, $(\mathcal{E}_1, \nabla_1) \in \mathrm{Ob}(\widehat{\textsc{Pic}}_{q^2}(\mathcal{O}_q(\mathbf{CP}^1)))$, so that

$$\forall m \in \mathbf{Z}, \qquad \Phi^m_{[\mathcal{E}_1, \nabla_1]}(\mathbf{F}_{[(\mathcal{E}_1, \nabla_1)]}) = q^{-2m} \mathbf{F}_{[(\mathcal{E}_1, \nabla_1)]},$$

$$\forall m \in \mathbf{Z}, \qquad \mathbf{F}_{[\mathcal{E}_m, \nabla_m)]} = [m]_{q^{-2}} \mathbf{F}_{[(\mathcal{E}_1, \nabla_1)]}.$$

# Clifford representations

## Definition

A *Clifford representation* $(H, \pi)$ of $\Omega_B$ consists of:

1. a $\mathbf{Z}_2$-graded separable Hilbert space $H$ equipped with a faithful $*$-representation $B \longrightarrow \mathbf{L}^{\mathrm{even}}(H)$;
2. an injective $*$-preserving $B$-bimodule morphism $\pi : \Omega^1_B \longrightarrow \mathbf{L}^{\mathrm{odd}}(H)$.

## Example

A spectral triple $(B, H, D)$ *spatially implements* $\Omega_B$ whenever the following defines a Clifford representation $(H, \pi_D)$ of $\Omega_B$:

$$\forall b \in B, \qquad \pi_D(\mathrm{d}_B(b)) := [D, b];$$

in this case, we call $(H, \pi_D)$ the Clifford representation induced by $(B, H, D)$.

# Equicontinuity

## Definition (Ć., cf. Bellissard–Marcolli–Reihani '10)

Let $L \in \textsc{Hom}^{\mathrm{bar}}(\mathbf{Z}, \widehat{\textsc{Pic}}(B))$; for each $k \in \mathbf{Z}$, let $(\mathcal{L}_k, (\sigma_k, \nabla_k)) := L_k$. We call $R$ *equicontinuous* with respect to a Clifford representation $(H, \pi)$ of $\Omega_B$ if

$$\forall \beta \in \Omega_B^1, \quad \sup_{k \in \mathbf{Z}} \sup_{\xi \in \mathcal{L}_k \setminus \{0\}} \frac{\|(\mathrm{id} \otimes \pi) \circ \sigma_k(\beta \otimes \xi)\|_{\mathcal{L}_k \otimes_B \mathbf{L}(H)}}{\|\xi\|_{\mathcal{L}_k}} < +\infty.$$

Hence:

1. $(P, \Omega_{P,\mathrm{h}}) \in \mathrm{Ob}(\widehat{\textsc{Circ}}(B))$ is *equicontinuous* w.r.t. $(H, \pi)$ if $\hat{\mathcal{L}}(P, \Omega_{P,\mathrm{h}})$ is;
2. $(E, \nabla_E) \in \mathrm{Ob}(\widehat{\textsc{Pic}}(B))$ is *equicontinous* w.r.t. $(H, \pi)$ if $\hat{\mathcal{P}}(E, \nabla_E)$ is;
3. given $\kappa > 0$, we say that $(P, \Omega_P, \Pi_P) \in \mathrm{Ob}(\textsc{Gauge}_\kappa(B))$ is *equicontinous* w.r.t. $(H, \pi)$ if $\hat{\mathcal{L}} \circ \mathrm{Hor}_\kappa(P, \Omega_P, \Pi_P)$ is.

Thus, equicontinuity w.r.t. $(H, \pi)$ is preserved by all relevant equivalences of categories and is stable under isomorphism.

# Examples

## Example (cf. Bellissard–Marcolli–Reihani '10)

Let $\phi \in \widehat{\mathrm{Aut}}(B)$. Then

$$\hat{\mathcal{P}}(\hat{\tau}_\phi) \cong \hat{\tau} \circ (k \mapsto \phi^k) \cong \hat{\mathcal{L}}(B \rtimes^{\mathrm{alg}}_{\phi} \mathbf{Z}, (\Omega_B \rtimes^{\mathrm{alg}}_{\phi} \mathbf{Z}, \mathrm{d}_\phi))$$

is equicontinuous w.r.t. a Clifford representation $(H, \pi)$ of $\Omega_B$ iff

$$\forall b \in B, \qquad \sup_{k \in \mathbf{Z}} \|\pi \circ \mathrm{d} \circ \phi^k(b)\|_{\mathbf{L}(H)} < +\infty.$$

## Example (Ć.)

Let $q \in \mathbf{R} \setminus \{0, \pm 1\}$. Then the $q$-monopole

$$(\mathcal{O}_q(\mathrm{SU}_2), \Omega_{q,3}(\mathrm{SU}_2), \Pi_q) \in \mathrm{Ob}(\mathrm{GAUGE}_{q^2}(B))$$

is *not* equicontinuous w.r.t. the Clifford representation of $\Omega_q(\mathbf{CP}^1)$ induced by the canonical spectral triple on $\mathcal{O}_q(\mathbf{CP}^1)$ of Dąbrowski–Sitarz.

# Lifting Clifford representations

## Definition (Ć.)

Let $\kappa > 0$, let $(P, \Omega_P, \Pi_P) \in \mathrm{Ob}(\textsc{Gauge}_\kappa(B))$, and let $(H, \pi)$ be a Clifford representation of $\Omega_B$. A *lift* of $(H, \pi)$ w.r.t. $(P, \Omega_P, \Pi_P)$ is $(\widetilde{H}, V, \upsilon, \widetilde{\pi})$, where:

1. $\widetilde{H}$ is a $\mathbf{Z}_2$-graded separable Hilbert space equipped with a strongly continuous unitary representation $\mathrm{U}(1) \to \mathrm{U}^{\mathrm{even}}(H)$ and a faithful $\mathrm{U}(1)$-equivariant $*$-representation $P \to \mathbf{L}^{\mathrm{even}}(H)$, such that
$$\overline{P \cdot H^{\mathrm{U}(1)}}^H = H, \quad \{p \in P \mid p\restriction_{H^{\mathrm{U}(1)}} = 0\} = \{0\};$$
2. $V$ is a separable $\mathbf{Z}_2$-graded Hilbert space and $\upsilon : V \mathbin{\widehat{\otimes}} H \to H^{\mathrm{U}(1)}$ is unitary;
3. $\widetilde{\pi} : \Omega^1_P \to \mathrm{End}^{\mathrm{odd}}_{\mathbf{C}}(P \cdot H^{\mathrm{U}(1)})$ is a faithful $*$-preserving $\mathrm{U}(1)$-equivariant $P$-bimodule morphism, such that
$$\forall \beta \in \Omega^1_B, \forall \xi \in H^{\mathrm{U}(1)}, \qquad \widetilde{\pi}(\beta)\xi = \upsilon(\mathrm{id} \mathbin{\widehat{\otimes}} \pi(\beta))\upsilon^* \xi.$$

We call $(\widetilde{H}, V, \upsilon, \widetilde{\pi})$ *bounded* whenever $\widetilde{\pi}(\Omega^1_P) \subset \mathbf{L}^{\mathrm{odd}}(\widetilde{H})$.

# Existence of bounded lifts

## Theorem (Ć.)

*Suppose that we are given:*

1. *a Clifford representation* $(H, \pi)$ *of* $\Omega_B$*;*
2. $\kappa > 0$ *and* $(P, \Omega_P, \Pi_P) \in \mathrm{Ob}(\textsc{Gauge}_\kappa(P))$.

*If* $(H, \pi)$ *admits a bounded lift with respect to* $(P, \Omega_P, \Pi_P)$*, then:*

1. $(P, \Omega_P, \Pi_P)$ *is equicontinuous with respect to* $(H, \pi)$*;*
2. $\kappa = 1$.

## Corollary (Ć.)

*Let* $q \in \mathbf{R} \setminus \{0, \pm 1\}$*, and let* $(H, \pi)$ *be a Clifford representation of* $\Omega_q(\mathbf{CP}^1)$*, e.g., the one induced by the canonical spectral triple on* $\mathcal{O}_q(\mathbf{CP}^1)$ *of Dąbrowski–Sitarz. Then* $(H, \pi)$ *does not admit a bounded lift with respect to the q-monopole*

$$(\mathcal{O}_q(\mathrm{SU}_2), \Omega_{q,3}(\mathrm{SU}_2), \Pi_q) \in \mathrm{Ob}(\textsc{Gauge}_{q^2}(B)).$$