Gauge theory on NC principal bundles

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Shameless self-promotion

References

B. Ć. and B. Mesland, *Gauge theory on noncommutative Riemannian principal bundles*, `arXiv:1912.04179`

B. Ć., *Non-trivial gauge theory on cleft quantum principal bundles*, (in preparation)

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Note

Today, we specialise to unital NC principal $\mathrm{U}(1)$-bundles with totally geodesic orbits of unit length.

Basic setup

Let $G = U(1)$, so that:

- $d\mu(z) \coloneqq \frac{1}{2\pi i}\frac{dz}{z}$ is the normalised Haar measure;
- $\mathfrak{g}^* = \mathbb{R}dt$ for $dt \coloneqq -i\frac{dz}{z}$ and $\mathfrak{g} = \mathbb{R}\frac{\partial}{\partial t}$ for $(dt, \frac{\partial}{\partial t}) \coloneqq 1$.

Thus, a unital G-C^*-algebra (A, α) is *principal* iff

$$\forall n \in \mathbb{Z}, \quad \overline{A_n^* \cdot A_n} = A^G, \quad A_n \coloneqq \{a \in A \mid \forall z \in G,\ \alpha_z(a) = z^n a\},$$

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Example

The trivial case $A = A^G \rtimes \mathbb{Z} \hookleftarrow A^G$, where $\mathbb{Z} \cong \widehat{G}$.

Example (Matsumoto, cf. Brzeziński–Sitarz)

The θ-deformed $\mathbb{C}$-Hopf fibration $C(S^3_\theta) \hookleftarrow C(S^3_\theta)^G \cong C(S^2)$.

Equivariant spectral triples

Let (A, α) be a unital sep'ble G-C^*-algebra. *Let $n \geqslant 1 = \dim G$.*

An *n-multigraded* G-*spectral triple* for (A, α) is $(\mathcal{A}, H, D, U)$:

1. (H, U) is a faithful $\mathbb{Z}_2$-graded covariant $*$-representation of $(\mathbb{C}l_n \widehat{\otimes} A, \mathrm{id} \widehat{\otimes} \alpha)$;
2. D is an odd G-invariant self-adjoint operator on H s.t.

$$(D + i)^{-1} \in K(H), \quad [D, \mathbb{C}l_n] = \{0\}, \quad \mathrm{Dom}(D) \subset C^1(H, U);$$

3. $\mathcal{A} \subset A$ is a dense G-invariant $*$-subalgebra s.t.

$$\mathcal{A} \subset C^1(A, \alpha), \quad \mathcal{O}(G) * \mathcal{A} \subseteq \mathcal{A}, \quad [D, \mathcal{A}] \subset B(H).$$

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Example

$(\mathcal{O}(G), L^2(G)^{\oplus 2}, \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \frac{\partial}{\partial t}, \text{translation})$ for $(C(G), \text{translation})$.

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Points 1 and 2 hint at possibilities for NC *gauge theory*.

Definition (cf. Dąbrowski–Sitarz, Forsyth–Rennie)

A *vertical geometry* for $(\mathcal{A}, H, D, U)$ is odd $c(dt) \in B(H)^G$, s.t.

1. $c(dt)^* = -c(dt)$ and $c(dt)^2 = -4\pi^2$,
2. $[c(dt), \mathbb{C}l_n] = [c(dt), \mathcal{A}] = \{0\}$,
3. $\mu(\frac{\partial}{\partial t}) \coloneqq -\frac{1}{2}[D, \frac{1}{4\pi^2}c(dt)] - dU(\frac{\partial}{\partial t}) \in B(H)$.

Vertical geometries and vertical Dirac operators

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We also define $V_1\mathcal{A} \coloneqq \mathbb{C}l_1 \cdot \mathbb{C}[c(dt)] \cdot \mathcal{A}$ and $V_1A \coloneqq \overline{V_1\mathcal{A}}^{B(H)}$.

Remainders and horizontal Dirac operators

Definition

A *remainder* for $(\mathcal{A}, \mathrm{H}, \mathrm{D}, \mathrm{U}; c(dt))$ is $Z \in B(H)^G$ odd, s.t.

$$Z^* = Z, \quad [Z, \mathbb{C}l_n] = 0.$$

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Example

The *canonical remainder* for $(\mathcal{A}, H, D; U; c(dt))$ is

$$Z_{\mathrm{can}} \coloneqq c(dt)\mu(\tfrac{\partial}{\partial t}).$$

Principal spectral triples

Definition

If (A, α) is principal, then $(\mathcal{A}, H, D, U; c(dt); Z)$ defines a *principal* G*-spectral triple* for (A, α) whenever

1. $\forall n \in \mathbb{Z}, \quad \overline{(V_1\mathcal{A})_n \cdot H^G} = H_n$,
2. $\{\omega \in V_1\mathcal{A} \mid \omega|_{H^G} = 0\} = \{0\}$,
3. $[D_h[Z], \mathcal{A}] \subset \overline{A \cdot [D - Z, \mathcal{A}^G]}^{B(H)}$,
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Examples (cf. Brain–Mesland–Van Suijlekom)

1. The canonical spectral triple for $C(\mathbb{T}^2_\theta) \cong C(\mathbb{T}) \rtimes_\theta \mathbb{Z}$.
2. The canonical spectral triple for $C(S^3_\theta)$.

Analysis

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1. $c(dt)$ encodes orbitwise intrinsic geometry and index theory via the *wrong-way cycle* (cf. Wahl)

$$(\mathcal{A}, E_1, S_1, V_1) \coloneqq (\mathcal{A}, L^2(V_1 A, \mathbb{E}_{V_1 A^G}), c(dt)d\alpha(\tfrac{\partial}{\partial t}), \alpha);$$

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Note (cf. Carey–Neshveyev–Nest–Rennie, Arici–Kaad–Landi...)

Since $G = U(1)$, the cycle $(\mathcal{A}, E_\rho, S_\rho)$ represents the extension class $[\partial] \in KK_1(A, A^G)$ of A as a Pimsner algebra.

Synthesis

Theorem

Let $(\mathcal{A}, H, D, U; c(dt); Z)$ *be a principal* G*-spectral triple:*

1. $H \cong E_1 \widehat{\otimes}_{V_1 A^G} H^G$ *and* $D_v = S_1 \widehat{\otimes} \operatorname{id}$;
2. $[D_h[Z], \cdot]$ *canonically induces a Hermitian connection* $\nabla[Z]$ *on* E_1 *s.t.* $D_h[Z] = \operatorname{id} \widehat{\otimes}_{\nabla[Z]} D^G[Z]$;
3. $[D] = [S_1] \otimes_{V_1 A^G} [D^G[Z]]$ *in* G*-equivariant* KK*-theory.*

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Thus, in G-equivariant unbounded KK-theory,

$$(\mathcal{A}, H, D - Z, U) \\ \cong (\mathcal{A}, E_1, S_1, V_1; \nabla[Z]) \widehat{\otimes}_{V_1 \mathcal{A}^G} (V_1 \mathcal{A}^G, H^G, D^G[Z], \mathrm{id}).$$

Gauge potentials

Fix a principal G-sp. tr. $(\mathcal{A}, H, D_0, U; c(dt); 0)$ for (A, α).

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Definition

A *gauge potential* is an operator D on H s.t.

1. $(\mathcal{A}, H, D, U; c(dt); 0)$ is a principal G-sp. tr. for (A, α),
2. $(D - D_0)(D_v + i)^{-1} \in B(H)$,
3. $[D - D_0, \mathcal{A}^G] = \{0\}$ and $[D - D_0, c(dt)] = 0$.

Let $\mathfrak{At}$ be the set of all gauge potentials.

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Let $\mathfrak{At}$ be the set of all gauge potentials.

It follows that for all $D, D' \in \mathfrak{At}$,

$$[D] = [S_1] \widehat{\otimes}_{V_1 A^G} [D^G[0]] = [S_1] \widehat{\otimes}_{V_1 A^G} [(D')^G[0]] = [D'].$$

Relative gauge potentials

Definition

A *relative gauge potential* is an odd operator $\mathbb{A}$ on H, s.t.

1. for some (and hence every) $D \in \mathfrak{At}$, we have

$$[\mathbb{A}, \mathcal{A}] \subset \overline{A \cdot [D, \mathcal{A}^G]}^{B(H)},$$

2. $\mathbb{A}(D_v + i)^{-1} \in B(H)$,
3. $[\mathbb{A}, \mathbb{C}l_n] = [\mathbb{A}, \mathcal{A}^G] = \{0\}$ and $[\mathbb{A}, c(dt)] = 0$;

let $\mathfrak{at}$ be the $\mathbb{R}$-vector space of all relative gauge potentials.

Thus, for all $D_1, D_2 \in \mathfrak{At}$, we have $D_1 - D_2 \in \mathfrak{at}$.

Gauge transformations

Definition

A *gauge transformation* is $\mathbb{S} \in \mathrm{U}(\mathrm{H})^{\mathrm{G}}$ even, s.t.

1. $\mathbb{S}\mathcal{A}\mathbb{S}^* \subseteq \mathcal{A}$,
2. $[\mathbb{S}, \mathbb{C}\mathrm{l}_n] = [\mathbb{S}, \mathcal{A}^{\mathrm{G}}] = \{0\}$ and $[\mathbb{S}, c(dt)] = 0$,
3. for some (and hence every) $\mathrm{D} \in \mathfrak{At}$, we have $\mathbb{S}[\mathrm{D}, \mathbb{S}^*] \in \mathfrak{at}$;

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3. for some (and hence every) $\mathrm{D} \in \mathfrak{At}$, we have $\mathbb{S}[\mathrm{D}, \mathbb{S}^*] \in \mathfrak{at}$;

let $\mathfrak{G}$ be the group of all gauge transformations.

We can now define the *gauge action* of $\mathfrak{G}$ on $\mathfrak{At}$ by

$$\forall \mathbb{S} \in \mathfrak{G}, \ \forall \mathrm{D} \in \mathfrak{At}, \quad \mathbb{S} \triangleright \mathrm{D} := \mathbb{S}\mathrm{D}\mathbb{S}^* \in \mathfrak{At}$$

and the *gauge action* of $\mathfrak{G}$ on $\mathfrak{at}$ by

$$\forall \mathbb{S} \in \mathfrak{G}, \ \forall \mathbb{A} \in \mathfrak{at}, \quad \mathbb{S} \triangleright \mathbb{A} := \mathbb{S}\mathbb{A}\mathbb{S}^* \in \mathfrak{at}.$$

A punchline of sorts

Theorem

1. *$\mathfrak{At}$ is a $\mathbb{R}$-affine space with space of translations $\mathfrak{at}$.*
2. *The gauge action of $\mathfrak{G}$ on $\mathfrak{At}$ is affine with linear part the gauge action of $\mathfrak{G}$ on $\mathfrak{at}$.*

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Example

The commutative case (up to an explicit groupoid cocycle).

Example

For the canonical spectral triple on $C(\mathbb{T}) \rtimes_\theta \mathbb{Z} \cong C(\mathbb{T}^2_\theta)$,

$$\{\mathbb{A} \in \mathfrak{at} \mid \mathbb{A}|_{\mathrm{H}^G} = 0\} \cong Z^1(\mathbb{Z}, C(\mathbb{T}, \mathbb{R})),$$
$$\{\mathbb{S} \in \mathfrak{G} \mid \mathbb{S}|_{\mathrm{H}^G} = \mathrm{id}\} \cong Z^1\left(\mathbb{Z}, C^1(\mathbb{T}, U(1))\right),$$

with $\mathfrak{s} \triangleright (\text{basepoint} + \mathfrak{a}) = \text{basepoint} + (\mathfrak{a} + \mathfrak{s} d \mathfrak{s}^*)$.