Quantum principal $\mathrm{U}(1)$-bundles
Differential, Riemannian, and metric geometry

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Shamless self-promotion

B. Ć., *Geometric foundations for classical* $\mathrm{U}(1)$*-gauge theory on noncommutative manifolds*, `arXiv:2301.01749`.

B. Ć. and B. Mesland, *Gauge theory on noncommutative Riemannian principal bundles*, Commun. Math. Phys. 388, 107-198 (2021).

B. Ć. and T. Venkata Karthik, *Euclidean Maxwell's equations on noncommutative Riemannian manifolds*, in preparation.

Existence of electromagnetic potentials

Problem

Let M be a closed manifold; Let $F \in \Omega^2(M, \mathbf{R})$ be closed.

Find $A \in \Omega^1(M, \mathbf{R})$ satisfying $F = \mathrm{d}A$.

Global solutions exist IFF $[F] = 0$ in $H^2(M, \mathbf{R})$.

A *local* solution A on U defines a connection $\mathrm{d} + \mathrm{i}A$ on the trivial line bundle $U \times \mathbf{C}$ with curvature $-\mathrm{i}(\mathrm{d} + \mathrm{i}A)^2 = F\restriction_U$.

Modified Problem (Weyl, Fock, London)

Even if $[F] \neq 0$, find a Hermitian line bundle with connection (L, ∇) on M with curvature $-\mathrm{i}\nabla^2 = F$.

Dirac quantisation

Theorem (Kobayashi, Weil, Kostant)

The following are equivalent.

1. *There exists a Hermitian line bundle with connection (L, ∇) on M with curvature F.*
2. *The 2-form F is integral, i.e., $\int_\Sigma F \in 2\pi\mathbf{Z}$ for every closed surface S in M.*

Example

The *Fubini–Study form* $\omega_{\mathrm{FS}} \coloneqq \frac{1}{(1+z\bar{z})^2}\,\mathrm{d}\,z \wedge \mathrm{d}\,\bar{z}$ on $\mathbf{CP}^1$ satisfies $[\omega_{\mathrm{FS}}] \neq 0$.

The 2-form $2\omega_{FS}$ has integral periods since $\int_{\mathbf{CP}^1} 2\omega_{\mathrm{FS}} = 2\pi$.

The *Chern connection* ∇ on $\mathcal{O}(-1) \to \mathbf{CP}^1$ has curvature $2\omega_{\mathrm{FS}}$.

Principal $\mathrm{U}(1)$-bundles with connection

Theorem

1. *If* (P, θ) *is a principal* $\mathrm{U}(1)$*-bundle with connection on* M*, then the irrep* $-\mathbf{1} := (\mathbf{C}, z \mapsto z^{-1}\,\mathrm{id})$ *yields the associated Hermitian line bundle with connection* $(P \times_{-\mathbf{1}} \mathbf{C}, \mathrm{d}^{\theta})$.
2. *If* (L, ∇) *is a Hermitian line bundle with connection on* M*, then there exists an essentially unique principal* $\mathrm{U}(1)$*-bundle with connection* (P, θ) *on* M*, such that* $(P \times_{-\mathbf{1}} \mathbf{C}, \mathrm{d}^{\theta}) \cong (L, \nabla)$.

Example

Give $\mathbf{S}^3 \subset \mathbf{C}^2$ the diagonal action of $\mathrm{U}(1)$; let $\theta_{\mathrm{Dirac}} := \overline{z_1}\,\mathrm{d}z_1 + \overline{z_2}\,\mathrm{d}z_2$.

The *Hopf fibration* with *Dirac monopole* connection $(\mathbf{S}^3, \theta_{\mathrm{Dirac}})$ defines the essential unique principal $\mathrm{U}(1)$-bundle with connection on $\mathbf{CP}^1$, such that

$$(\mathbf{S}^3 \times_{-\mathbf{1}} \mathbf{C}, \mathrm{d}^{\theta_{\mathrm{Dirac}}}) \cong (\mathcal{O}(-1), \nabla).$$

Why care about the noncommutative case?

Sales Pitch

Noncommutative geometry permits the conceptually economical (semi-classical) modelling of quantum physics as classical physics on noncommutative manifolds.

Aspirational Example (cf. Majid et al.)

Approximate quantum gravity coupled with $\mathrm{U}(1)$-gauge theory.

Aspirational Example (cf. Nekrasov–Schwarz, Peterka)

Construct noncommutative $\mathrm{U}(1)$-instantons where they don't exist commutatively.

Aspirational Example (cf. Avron–Seiler–Zograf, Hannabuss–Mathai–Thiang)

Broaden the mathematical toolkit for physics around the quantum Hall effect.

The quantum 3-sphere

Fix $q \in (0,1)$.

Definition (Woronowicz, Vaksman–Sojbel'man)

The *quantum 3-sphere* is the universal unital pre-C*-algebra $\mathcal{O}_q(\mathbf{S}^3)$ with generators z_0 and z_1 and relations

$$z_0 z_1 = q z_1 z_0, \quad z_0 z_1^* = q z_1^* z_0, \quad z_1^* z_1 = z_1 z_1^*,$$
$$z_0^* z_0 + z_1^* z_1 = 1, \quad z_0 z_0^* + q^2 z_1 z_1^* = 1.$$

Heuristic

At $q = 1$, recover the dense $*$-subalgebra of spherical harmonics in $C(\mathbf{S}^3)$.

Question

How do we make $\mathcal{O}_q(\mathbf{S}^3)$ into a noncommutative (Riemannian) manifold or quantum metric space?

The quantum projective line

Say that a $\mathsf{U}(1)$-action on a unital pre-C^*-algebra P is *finite-type* whenever

$$P = \bigoplus_{m \in \mathbf{Z}} P_m, \qquad P_m := \{p \in P \mid \forall w \in \mathsf{U}(1),\ w \triangleright p = w^m p\}.$$

Definition (Podleś)

The *quantum projective line* is $\mathcal{O}_q(\mathbf{CP}^1) := \mathcal{O}_q(\mathbf{S}^3)^{\mathsf{U}(1)}$, where $\mathcal{O}_q(\mathbf{S}^3)$ has the unique finite-type $\mathsf{U}(1)$-action satisfying $\{z_0, z_1\} \subset \mathcal{O}_q(\mathbf{S}^3)_1$.

Heuristic

At $q = 1$, recover the dense $*$-subalgebra of spherical harmonics in $C(\mathbf{CP}^1)$.

Goal

Lift geometry from $\mathcal{O}_q(\mathbf{CP}^1)$ to $\mathcal{O}_q(\mathbf{S}^3)$.

The quantum Hopf fibration

***Definition** (cf. Năstăsescu–Van Ostaeyen, Sitarz–Venselaar, Arici–Kaad–Landi)*

Let P be a unital pre-C^*-algebra with a finite-type $\mathrm{U}(1)$-action. Then P is a *topological quantum principal* $\mathrm{U}(1)$*-bundle* over $P^{\mathrm{U}(1)}$ whenever:

1. there exists $(e_i)_{i=1}^m$ in P_1 satisfying $\sum_{i=1}^m e_i e_i^* = 1$;
2. there exists $(f_j)_{j=1}^n$ in P_1 satisfying $\sum_{j=1}^n f_j^* f_j = 1$.

***Example** (Brzeziński–Majid)*

The *quantum Hopf fibration* is the topological quantum principal $\mathrm{U}(1)$-bundle $\mathcal{O}_q(\mathbf{S}^3)$ over $\mathcal{O}_q(\mathbf{CP}^1)$.

Proof

1. The family (z_0, qz_1) in $\mathcal{O}_q(\mathbf{S}^3)_1$ satisfies $z_0 z_0^* + (qz_1)(qz_1)^* = 1$.
2. The family (z_0, z_1) in $\mathcal{O}_q(\mathbf{S}^3)_1$ satisfies $z_0^* z_0 + z_1^* z_1 = 1$. □

The quantum Hopf line bundle

Definition (Rieffel; cf. Kajiwara–Watatani)

Let B be a unital pre-C*-algebra. A *Hermitian line B-bimodule* is a B-bimodule E with left, right B-valued inner products ${}_E(\cdot,\cdot)$, $(\cdot,\cdot)_E$, with:

1. $\|{}_E(xb,xb)\| \leqslant \|b\|^2\|{}_E(x,x)\|$ and $\|(bx,bx)_E\| \leqslant \|b\|^2\|(x,x)_E\|$;
2. ${}_E(x,yb) = {}_E(xb^*,y)$ and $(x,by)_E = (b^*x,y)_E$;
3. there exist $(e_i)_{i=1}^m$, $(f_j)_{j=1}^n$ in E with $\sum_{i=1}^m {}_E(e_i,e_i) = 1 = \sum_{j=1}^n (f_j,f_j)_E$;
4. ${}_E(x,y)\cdot z = x\cdot(y,z)_E$.

Example (Hajac–Majid)

Each $\mathcal{L}_m \coloneqq \mathcal{O}_q(\mathbf{S}^3)_m$ defines a Hermitian line $\mathcal{O}_q(\mathbf{CP}^1)$-bimodule with respect to ${}_{\mathcal{L}_m}(x,y) \coloneqq xy^*$ and $(x,y)_{\mathcal{L}_m} \coloneqq x^*y$.

In particular, $\mathcal{L}_1 \coloneqq \mathcal{O}_q(\mathbf{S}^3)_1$ is the *quantum Hopf line bundle*.

Pimsner's construction

***Theorem** (Arici–Kaad–Landi; cf. Pimsner, Abadie–Eilers–Exel, Beggs–Brzeziński)*

Let B be a unital pre-C^*-algebra.

1. *Let P be a topological quantum principal $\mathrm{U}(1)$-bundle over B. Then P_1 is a Hermitian line B-bimodule with ${}_{P_1}(x,y) \coloneqq xy^*$, $(x,y)_{P_1} \coloneqq x^*y$.*
2. *Let E be a Hermitian line B-bimodule. There exists an essentially unique topological quantum principal $\mathrm{U}(1)$-bundle P over B with $P_1 \cong E$.*

Example

The quantum Hopf fibration $\mathcal{O}_q(\mathbf{S}^3)$ is the essentially unique topological quantum principal $\mathrm{U}(1)$-bundle over B satisfying $\mathcal{O}_q(\mathbf{S}^3)_1 \cong \mathcal{L}_1$.

Strategy

Use the quantum Hopf line bundle $\mathcal{L}_1$ to lift geometry from $\mathcal{O}_q(\mathbf{CP}^1)$ to $\mathcal{O}_q(\mathbf{S}^3)$ *qua* total space of the quantum Hopf fibration.

Pimsner's construction unpacked

Let B be a unital pre-C^*-algebra.

Theorem (Rieffel, Brown–Green–Rieffel, Buss–Meyer–Zhu)

The Hermitian line B-bimodules form a coherent 2-group $\mathrm{Pic}(B)$.

Proposition (d'après Joyal–Street)

For every object g of a coherent 2-group G*, there exists an essentially unique weak monoidal functor $F : \mathbf{Z} \to \mathsf{G}$, such that $F(1) \cong g$.*

Theorem (Ć.)

Every weak monoidal functor between coherent 2-groups canonically defines a bar functor à la Beggs–Majid (involutive functor à la Egger).

Corollary (Buss–Meyer–Zhu, Schwieger–Wagner)

A weak monoidal functor from a group Γ to $\mathrm{Pic}(B)$ *defines a saturated pre-Fell bundle over Γ with fibre B over e.*

The quantum enveloping algebra of $\mathfrak{su}(2)$

Definition (Kuliš–Rešetihin)

The *quantum enveloping algebra* of $\mathfrak{su}(2)$ is the unital $*$-algebra $\mathcal{U}_q(\mathfrak{su}(2))$ with generators E, F, K and K^{-1} and relations

$$KK^{-1} = K^{-1}K = 1, \quad KEK^{-1} = qE, \quad KFK^{-1} = q^{-1}F,$$
$$EF - FE = \frac{1}{q-q^{-1}}(K^2 - K^{-2}),$$
$$E^* = F, \quad F^* = E, \quad K^* = K, \quad (K^{-1})^* = K^{-1}.$$

Definition (Skljanin)

A representation ∂ of $\mathcal{U}_q(\mathfrak{su}(2))$ on a unital $*$-algebra P is a *Hopf action* if:

$$\partial_K(xy) = \partial_K(x)\partial_K(y);$$
$$\partial_X(xy) = \partial_X(x)\partial_K(y) + \partial_{K^{-1}}(x)\partial_X(y), \quad X \in \{E, F\};$$
$$\partial_K(x^*) = \partial_{K^{-1}}(x)^*, \quad \partial_E(x^*) = -q\partial_F(x)^*, \quad \partial_F(x^*) = -q^{-1}\partial_E(x)^*.$$

The translation action on $\mathcal{O}_q(\mathbf{S}^3)$

Theorem (Masuda–Mimachi–Nakagami–Noumi–Saburi–Ueno)

There exists a unique Hopf action ∂ of $\mathcal{U}_q(\mathfrak{su}(2))$ on $\mathcal{O}_q(\mathbf{S}^3)$, such that

$$
\begin{aligned}
\partial_K(z_0) &= q^{-1/2} z_0, & \partial_K(z_1) &= q^{-1/2} z_1,\\
\partial_E(z_0) &= 0, & \partial_E(z_1) &= 0,\\
\partial_F(z_0) &= -q z_1^*, & \partial_F(z_1) &= z_0^*.
\end{aligned}
$$

Moreover, for all $m \in \mathbf{Z}$,

$$
\partial_K\restriction_{\mathcal{L}_m} = q^{-m/2}\,\mathrm{id}_{\mathcal{L}_m}, \qquad \partial_E(\mathcal{L}_m) \subseteq \mathcal{L}_{m-2}, \qquad \partial_F(\mathcal{L}_m) \subseteq \mathcal{L}_{m+2}.
$$

Heuristic

"By setting $K = q^{H/2}$ and taking $q \nearrow 1$, we recover the infinitesimal right translation action of $\mathfrak{su}(2)$ on $\mathrm{SU}(2) \cong \mathbf{S}^3$."

Differential calculus on $\mathcal{O}_q(\mathbf{CP}^1)$

***Theorem** (Podleś; cf. Majid, Landi–Reina–Zampini, Khalkhali–Landi–Van Suijlekom)*

There exists an essentially unique 2-dimensional left $\mathcal{O}_q(\mathrm{SU}(2))$-covariant $*$-exterior algebra $(\Omega_q(\mathbf{CP}^1), \mathrm{d})$ on $\mathcal{O}_q(\mathbf{CP}^1)$.

1. As a $\mathcal{O}_q(\mathbf{CP}^1)$-bimodule, let $\Omega_q(\mathbf{CP}^1) \coloneqq \bigoplus_{k=0}^{2} \Omega_q^k(\mathbf{CP}^1)$, where
$$\Omega_q^0(\mathbf{CP}^1) \coloneqq \mathcal{O}_q(\mathbf{CP}^1) \eqqcolon \Omega_q^2(\mathbf{CP}^1), \qquad \Omega_q^1(\mathbf{CP}^1) \coloneqq \mathcal{L}_{-2} \oplus \mathcal{L}_2.$$
2. Define multiplication, $*$ on $\Omega_q^1(\mathbf{CP}^1)$ by
$$(\omega_+, \omega_-) \cdot (\eta_+, \eta_-) \coloneqq \mathrm{i}(\omega_+\eta_- - q^{-2}\omega_-\eta_+), \quad * \coloneqq \begin{pmatrix} 0 & q^{-2}* \\ q^2* & 0 \end{pmatrix}.$$
3. Let $\partial_+ \coloneqq \partial_{q^{1/2}EK}$ and $\partial_- \coloneqq \partial_{q^{-1/2}FK}$ on $\mathcal{O}_q(\mathbf{S}^3) = \bigoplus_{m\in\mathbf{Z}} \mathcal{L}_m$.
4. Define $\mathrm{d} : \Omega_q(\mathbf{CP}^1) \to \Omega_q^{\bullet+1}(\mathbf{CP}^1)$ by
$$\mathrm{d}^{(0)} \coloneqq -\mathrm{i}\begin{pmatrix} \partial_+ \\ \partial_- \end{pmatrix}, \qquad \mathrm{d}^{(1)} \coloneqq \begin{pmatrix} -q^{-2}\partial_- & \partial_+ \end{pmatrix}.$$

The q-monopole connection

Theorem (Landi–Reina–Zampini, Khalkhali–Landi–Van Suijlekom)

The following constructs a Hermitian bimodule connection (σ_m, ∇_m) on $\mathcal{L}_m$.

1. Define $\mu_{j,k} : \mathcal{L}_j \otimes_{\mathcal{O}_q(\mathbf{CP}^1)} \mathcal{L}_k \xrightarrow{\cong} \mathcal{L}_{j+k}$ by $\mu_{j,k}(x \otimes y) \coloneqq xy$.
2. Define $\nabla_m : \mathcal{L}_m \otimes_{\mathcal{O}_q(\mathbf{CP}^1)} \Omega_q^\bullet(\mathbf{CP}^1) \to \mathcal{L}_m \otimes_{\mathcal{O}_q(\mathbf{CP}^1)} \Omega_q^{\bullet+1}(\mathbf{CP}^1)$ by
$$\nabla_m^{(0)} \coloneqq -\mathrm{i}\begin{pmatrix} \mu_{m,-2}^{-1} \circ \partial_+ \\ \mu_{m,2}^{-1} \circ \partial_- \end{pmatrix}, \quad \nabla_m^{(1)} \coloneqq \begin{pmatrix} -q^{-2}\partial_- \circ \mu_{m,-2} & \partial_+ \circ \mu_{m,2} \end{pmatrix}.$$
3. Recover $\sigma_m : \Omega_q^\bullet(\mathbf{CP}^1) \otimes_{\mathcal{O}_q(\mathbf{CP}^1)} \mathcal{L}_m \xrightarrow{\cong} \mathcal{L}_m \otimes_{\mathcal{O}_q(\mathbf{CP}^1)} \Omega_q^\bullet(\mathbf{CP}^1)$ as
$$\sigma_m^{(0)} \coloneqq \mu_{m,0}^{-1} \circ \mu_{0,m} \eqqcolon \sigma_m^{(2)}, \quad \sigma_m^{(1)} \coloneqq \begin{pmatrix} \mu_{m,-2}^{-1}\circ\mu_{-2,m} & 0 \\ 0 & \mu_{m,2}^{-1}\circ\mu_{2,m} \end{pmatrix}.$$

Theorem (Díaz García–Krutov–Ó Buachalla–Somberg–Strung)

The connection ∇_m is the unique left $\mathcal{O}_q(\mathrm{SU}(2))$-covariant connection on $\mathcal{L}_m$.

The differential Picard 2-group

Theorem (Ć.; cf. Beggs–Majid)

Let B be a unital pre-C^*-algebra with $*$-exterior algebra (Ω_B, d_B). The Hermitian line B-bimodules with connection form a coherent 2-group $\mathrm{DP{\scriptsize IC}}(B)$ with group of isomorphism classes $\mathrm{DPic}(B)$.

Example

If X is a closed manifold, then $\mathrm{DPic}(C^\infty(X)) \cong \check{H}^2(X) \rtimes \mathrm{Diff}(X)$.

Example (Ć.–Venkata Karthik; cf. Elliott, Kodaka, Nawata–Watatani)

If $\theta \in (0,1) \setminus \mathbf{Q}$ is Diophantine (e.g., algebraic), then

$$\mathrm{DPic}(C^\infty_\theta(\mathbf{T}^2)) \cong \begin{cases} \left(\frac{(\mathbf{R}^2)^* \times \mathbf{R}^2}{L_\theta} \rtimes \mathrm{SL}(2,\mathbf{Z})\right) \rtimes \mathbf{Z} & \text{if } \theta \text{ quadratic,} \\ \frac{(\mathbf{R}^2)^* \times \mathbf{R}^2}{L_\theta} \rtimes \mathrm{SL}(2,\mathbf{Z}) & \text{else,} \end{cases}$$

where $L_\theta := \langle (2\pi e_1^T; -2\pi\theta e_2), (2\pi e_2^T; 2\pi\theta e_1), (0; 2\pi e_1), (0; 2\pi e_2) \rangle$.

Curvature and its discontents

Let B be a unital pre-C*-algebra with $*$-exterior algebra (Ω_B, d_B).

Let (E, σ_E, ∇_E) be a Hermitian line B-bimodule with Hermitian bimodule connection; the *curvature* of $[E, \nabla_E]$ is closed central $\mathbf{F}_{[E,\nabla_E]} \in (\Omega^2_B)_{\mathrm{sa}}$ uniquely determined by $\nabla^2_E(e) = e \otimes \mathrm{i}\mathbf{F}_{[E,\nabla_E]}$.

Definition (Ć.)

Suppose that $\mathbf{F}_{[E,\nabla_E]} \neq 0$. The *vertical deformation parameter* of $[E, \nabla_E]$, if it exists, is $\kappa_{[E,\nabla_E]} \in \mathbf{R}^\times$ uniquely determined by

$$\sigma_E^{-1}(e \otimes \mathbf{F}_{[E,\nabla_E]}) = \kappa_{[E,\nabla_E]}\mathbf{F}_{[E,\nabla_E]} \otimes e.$$

Example (cf. Landi–Reina–Zampini, Khalkhali–Landi–Van Suijlekom)

Let $\mathrm{vol} := 1 \in \Omega^2_q(\mathbf{CP}^1)$; given $t \in (0,1)$, let $[n]_t := \frac{1-t^{n-1}}{1-t}$. Then

$$\mathbf{F}_{[\mathcal{L}_m,\nabla_m]} = [m]_{q^{-2}} q^2 \,\mathrm{vol}, \qquad \kappa_{[\mathcal{L}_m,\nabla_m]} = q^{2m}.$$

Pimsner's construction redux

Theorem (Ć.; cf. Đurđević, Moncada)

Let B be a unital pre-C^*-algebra with $*$-exterior algebra (Ω_B, d_B). Let $\kappa > 0$.

1. *If $(P; \Omega_P, \mathrm{d}_P; \Pi_P)$ is a κ-differentiable quantum principal $\mathrm{U}(1)$-bundle with connection over $(B; \Omega_B, \mathrm{d}_B)$, then P_1 admits a canonical Hermitian bimodule connection $(\sigma_{P_1}, \nabla_{P_1})$ with $\kappa_{[P_1, \nabla_{P_1}]} = \kappa$.*
2. *If $(E; \sigma_E, \nabla_E)$ is a Hermitian line B-bimodule with connection that satisfies $\kappa_{[E, \nabla_E]} = \kappa$, then there exists an essentially unique κ-differentiable quantum principal $\mathrm{U}(1)$-bundle with connection $(P; \Omega_P, \mathrm{d}_P; \Pi_P)$ over $(B; \Omega_B, \mathrm{d}_B)$, such that $(P_1; \sigma_{P_1}, \nabla_{P_1}) \cong (E; \sigma_E, \nabla_E)$.*

Example (cf. Đurđević)

$(\mathcal{O}_q(\mathbf{CP}^1); \Omega_q(\mathbf{CP}^1), \mathrm{d}_q)$ lifts via $(\mathcal{L}_1; \sigma_1, \nabla_1)$ to $\mathcal{O}_q(\mathbf{S}^3)$ with Woronowicz's *3-dimensional calculus* $(\Omega_q(\mathbf{S}^3), \mathrm{d}_q)$ and Brzeziński–Majid's *q-monopole connection* Π_q.

Lifting differential geometry to $\mathcal{O}_q(\boldsymbol{S}^3)$ via $(\mathcal{L}_1; \sigma_1, \nabla_1)$

1. Equip $\Omega_{q,\mathrm{h}}(\boldsymbol{S}^3) \coloneqq \mathcal{O}_q(\boldsymbol{S}^2) \otimes_{\mathcal{O}_q(\mathbf{CP}^1)} \Omega_q(\mathbf{CP}^1)$ with

$$(p_1 \otimes \beta_1) \cdot (p_2 \otimes \beta_2) \coloneqq p_1 \cdot \sigma_{|p_1|}(\beta_1 \otimes p_2) \cdot \beta_2,$$
$$(p \otimes \beta)^* \coloneqq \sigma_{-|p|}(\beta^* \otimes p^*).$$

2. Define $\mathrm{d}_{q,\mathrm{h}} : \Omega^\bullet_{q,\mathrm{h}}(\boldsymbol{S}^3) \to \Omega^{\bullet+1}_{q,\mathrm{h}}(\boldsymbol{S}^3)$ by

$$\mathrm{d}_{q,\mathrm{h}}(p \otimes \beta) \coloneqq \nabla_{|p|}(p) \cdot \beta + p \otimes \mathrm{d}_q \beta.$$

3. Construct $\Omega_q(\boldsymbol{S}^3)$ from $\Omega_{q,\mathrm{h}}(\boldsymbol{S}^3)$ by adjoining $\vartheta \in \Omega^1_q(\boldsymbol{S}^3)^{\mathrm{U}(1)}_{\mathrm{sa}}$ satisfying $\vartheta^2 = 0$ and $\vartheta \cdot (p \otimes \beta) = q^{-2|p|}(p \otimes \beta) \cdot \vartheta$.
4. Define $\mathrm{d}_q : \Omega^\bullet_q(\boldsymbol{S}^3) \to \Omega^{\bullet+1}_q(\boldsymbol{S}^3)$ by $\mathrm{d}_q \vartheta \coloneqq -\frac{q^4}{2\pi}\,\mathrm{vol}$ and

$$\mathrm{d}_q(p \otimes \beta) \coloneqq \vartheta \cdot 2\pi\mathrm{i}[|p|]_{q^2}(p \otimes \beta) + \mathrm{d}_{q,\mathrm{h}}(p \otimes \beta).$$

5. Define $\Pi_q : \Omega_q(\boldsymbol{S}^3) \to \Omega_{q,\mathrm{h}}(\boldsymbol{S}^3)$ by $\Pi_q\restriction_{\Omega_{q,\mathrm{h}}(\boldsymbol{S}^3)} \coloneqq \mathrm{id}$ and $\Pi_q(\vartheta) \coloneqq 0$.

Bounded commutator representations

Let B be a unital pre-C^*-algebra; let (Ω_B, d_B) be a $*$-exterior algebra on B.

Definition (Baaj–Julg, Connes, Schmüdgen)

A *bounded commutator representation* of $(B; \Omega_B, \mathrm{d}_B)$ is (H, π, D), where

1. H is a separable $\mathbf{Z}/2\mathbf{Z}$-graded pre-Hilbert space,
2. $\pi : B \to \mathbf{L}(H)_{\mathrm{even}}$ is a bounded $*$-homomorphism,
3. $D : H \to H$ is an odd symmetric $\mathbf{C}$-linear map,

such that there exists (necessarily unique) $\pi_D : \Omega^1_B \to \mathbf{L}(H)_{\mathrm{odd}}$ satisfying

$$\pi_D(\mathrm{d}_B(b)) = \mathrm{i}[D, \pi(b)].$$

It is *faithful* whenever π is isometric and π_D is injective.

The spin Dirac commutator representation of $\mathcal{O}_q(\mathbf{CP}^1)$

Proposition (Dąbrowski–Sitarz, Majid; cf. Neshveyev–Tuset)

The following constructs a faithful bounded commutator representation $(\mathcal{S}_q(\mathbf{CP}^1), \pi_q, \not{D}_q)$ *of* $(\mathcal{O}_q(\mathbf{CP}^1), \Omega_q(\mathbf{CP}^1), \mathrm{d}_q)$.

1. Equip $\mathcal{L}_{\mp 1}$ with the inner product $\langle s_1, s_2 \rangle := h_q(s_1^* s_2)$, where h_q is Woronowicz's *Haar state* on $\mathcal{O}_q(\mathbf{S}^3)$.
2. Let $\mathcal{S}_q(\mathbf{CP}^1) := \mathcal{L}_{-1} \oplus \mathcal{L}_1$ as a direct sum of pre-Hilbert spaces with the $\mathbf{Z}/2\mathbf{Z}$-grading $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.
3. Let $\pi_q : \mathcal{O}_q(\mathbf{CP}^1) \to \mathbf{L}(\mathcal{S}_q(\mathbf{CP}^1))_{\mathrm{even}}$ be the direct sum of left $\mathcal{O}_q(\mathbf{CP}^1)$-module structures on $\mathcal{L}_{-1}$ and $\mathcal{L}_1$.
4. Let $\not{D}_q := \begin{pmatrix} 0 & q^{-1}\partial_+ \\ q\partial_- & 0 \end{pmatrix}$; hence $\pi_{\not{D}_q}(\omega_+, \omega_-) = -\begin{pmatrix} 0 & q^{-2}\omega_+ \\ q^2\omega_- & 0 \end{pmatrix}$.

No-go for bounded commutator representations

Proposition (Ć.)

Let $\kappa > 0$, *let* $(P; \Omega_P, \mathrm{d}_P; \Pi)$ *be a* κ*-differentiable quantum principal* $\mathrm{U}(1)$*-bundle with connection, and let* (H, π, D) *be a bounded commutator representation of* $(P; \Omega_P, \mathrm{d}_P)$*. If* $\kappa \neq 1$*, then* $(\mathrm{id} - \Pi)(\Omega^1_P) \subseteq \ker \pi_D$.

Remark

This is an easy application of the existence of $(e_i)_{i=1}^m$ and $(f_k)_{j=1}^n$ in P_1 that satisfy $\sum_{i=1}^m e_i e_i^* = 1 = \sum_{j=1}^n f_j^* f_j$.

Corollary (Schmüdgen)

If (H, π, D) *is a bounded commutator representation of* $(\mathcal{O}_q(\mathbf{S}^3); \Omega_q(\mathbf{S}^3), \mathrm{d}_q)$*, then* $\pi_D = 0$.

Modular symmetries

Let $(P;\Omega_P,\mathrm{d}_P;\Pi)$ be a κ-differentiable quantum principal $\mathrm{U}(1)$-bundle over $(B;\Omega_B,\mathrm{d}_B)$.

Let (H,U,π,D) be a *locally bounded* ($\mathrm{U}(1)$-equivariant) commutator representation of $(P;\Omega_P,\mathrm{d}_P;\Pi)$.

A *modular symmetry* of (H,U,π,D) is even positive $\mathrm{U}(1)$-invariant invertible locally bounded N on H, such that

$$N\restriction_{H^{\mathrm{U}(1)}}=\mathrm{id},\qquad [N,\pi(B)]=\{0\},\qquad N\pi(P)N^{-1}=\pi(P).$$

Example

Given $t\in(0,\infty)$, let $\Lambda_t\coloneqq\bigoplus_{j\in\mathbf{Z}}t^{-j}\,\mathrm{id}_{H_j}$, where

$$H_j\coloneqq\{\xi\in H\mid\forall w\in\mathrm{U}(1),\ U_w\xi=w^m\xi\}.$$

No-go for twisted commutator representations

Theorem (Ć.)

Suppose that $\mathrm{Z}(B) = \mathbf{C}$ *and that there exist* $\eta \in \Omega^1_{P,\mathrm{h}} \setminus \{0\}$ *and* $t \in (0, \infty) \setminus \{\kappa\}$ *satisfying* $\eta \cdot p = t^{-|p|} p \cdot \eta$. *Suppose that* π *is injective and that* $\pi(P) \cdot H^{\mathrm{U}(1)}$ *is dense in* H. *If there exists a modular symmetry* N *of* (H, U, π, D) *that satisfies*

$$N \cdot \pi_D(\Omega^1_P) \cdot N \subseteq \mathbf{L}(H),$$

then $(\mathrm{id} - \Pi)(\Omega^1_P) \subseteq \ker \pi_D$ *or* $\pi_D(\eta) = 0$.

Corollary

Let (H, U, π, D) *be a faithful locally bounded commutator representation of* $(\mathcal{O}_q(\mathbf{S}^3), \Omega_q(\mathbf{S}^3), \mathrm{d}_q)$, *such that* $\pi(P) \cdot H^{\mathrm{U}(1)}$ *is dense in* H. *There exists no modular symmetry* N *of* (H, U, π, D) *that satisfies*

$$N \cdot \pi_D(\Omega^1_q(\mathbf{S}^3)) \cdot N \subseteq \mathbf{L}(H).$$

TL;DR

- We can formulate and solve the lifting problem for Riemannian structure in terms of Hodge star operators and compatible traces (cf. Kustermans–Murphy–Tuset, Majid, Zampini, Ó Buachalla, Moncada).
  - This involves a precise generalisation of *conformality* to Hermitian line bimodules with connection.
  - This recovers Zampini's Hodge star operator on $(\Omega_q(\boldsymbol{S}^3), \mathrm{d}_q)$.
- We can formulate and solve the compatible lifting problem for Riemannian structure in terms of faithful commutator representations.
  - In general, we need separate modular symmetries to 'damp' vertical and horizontal 1-forms.
  - 'Dampability' of horizontal 1-forms is governed by a precise generalisation of *metric equicontinuity* à Belissard–Marcolli–Reihani to Hermitian line bimodules with connection.
  - We obtain a large class of candidate compact quantum metric spaces that recovers Kaad–Kyed's construction for $\mathcal{O}_q(\boldsymbol{S}^3)$.