Quantum principal $\mathrm{U}(1)$-bundles

Analysis & synthesis

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May 6, 2024

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Approaches to quantum principal bundles

1. T. Brzeziński & S. Majid, *Quantum group gauge theory on quantum spaces*, Commun. Math. Phys. **157** (1993), no. 3, 591–638:

$$\Omega^1_{P,\mathrm{hor}} \coloneqq \ker(\mathrm{ver}_{\mathrm{BM}} : \Omega^1_P \to \Lambda^1_H \otimes P) = P \cdot \mathrm{d}(P^{\mathrm{co}H}) \cdot P.$$

2. M. Đurđević, *Geometry of quantum principal bundles II*, Rev. Math. Phys. **9** (1997), no. 5, 531–607:

$$\Omega_{P,\mathrm{hor}} \coloneqq \{\omega \in \Omega_P \mid \Delta_{\Omega_P}(\omega) \in \Omega_P \otimes H\} \supseteq P \cdot \Omega_P^{\mathrm{co}\Omega_H} \cdot P.$$

Progress towards theoretical synthesis

1. B. Ć., *Classical gauge theory on quantum principal bundles*, `arXiv:2108.13789`.
2. B. Ć., *Geometric foundations for classical* U(1)*-gauge theory on noncommutative manifolds*, `arXiv:2301.01749`.
3. A. Del Donno, E. Latini, T. Weber, *On the Đurđević approach to quantum principal bundles*, `arXiv:2404.07944`.

Basic definitions

Let $\alpha : \mathrm{U}(1) \to \mathrm{GL}(V)$ be a linear representation of $\mathrm{U}(1)$.

For each $m \in \mathbf{Z}$, define the m^{th} *isotypical component*

$$V_m := \{v \in V \mid \forall z \in \mathrm{U}(1),\ \alpha_z(v) = z^m v\}.$$

Assumption

We only consider the case where $V = \bigoplus_{m \in \mathbf{Z}} V_m$.

Given $\kappa > 0$, define $\mathrm{U}(1)$-equivariant $\Lambda_\kappa, \partial_\kappa : V \to V$ by

$$\Lambda_\kappa := \bigoplus_{m \in \mathbf{Z}} \kappa^m \,\mathrm{id}_{V_m}, \qquad \partial_\kappa := \bigoplus_{m \in \mathbf{Z}} 2\pi i [m]_\kappa \,\mathrm{id}_{V_m},$$

where $[m]_\kappa := \frac{1-\kappa^m}{1-\kappa}$ if $\kappa \neq 1$ and $[m]_\kappa = m$ if $\kappa = 1$.

Basic definitions

A *graded $*$-algebra* is an $\mathbf{N}_0$-graded unital $\mathbf{C}$-algebra Ω equipped with $\mathbf{C}$-antilinear $* : \Omega^\bullet \to \Omega^\bullet$, such that

$$1^* = 1, \qquad (\alpha^*)^* = \alpha, \qquad (\alpha\beta)^* = (-1)^{\deg(\alpha)\deg(\beta)}\beta^*\alpha^*.$$

A *$*$-quasi-differential graded algebra* (*$*$-quasi-DGA*) is a graded $*$-algebra Ω equipped with $\mathbf{C}$-linear $\nabla : \Omega^\bullet \to \Omega^{\bullet+1}$, such that:

$$\nabla(\alpha^*) = \nabla(\alpha)^*, \quad \nabla(\alpha\beta) = \nabla(\alpha)\beta + (-1)^{\deg(\alpha)}\alpha\nabla(\beta).$$

A *$*$-differential graded algebra* (*$*$-DGA*) is a $*$-quasi-DGA (Ω, d), such that $\mathrm{d}^2 = 0$.

A *$*$-exterior algebra* is a $*$-DGA (Ω, d), such that Ω is generated as a ring by Ω^0 and $\mathrm{d}(\Omega^0)$.

Invariant $*$-exterior algebras on $\mathcal{O}(\mathrm{U}(1))$

Consider $\mathcal{O}(\mathrm{U}(1)) = \mathbf{C}[z, z^{-1}]$ with

$$(z^m)^* = z^{-m}, \qquad \alpha_w(z^m) \coloneqq w^m z^m.$$

Given $\kappa > 0$, construct $(\Omega_\kappa(\mathrm{U}(1)), \mathrm{d})$ from $\Omega^0_\kappa(\mathrm{U}(1)) \coloneqq \mathcal{O}(\mathrm{U}(1))$ by appending $e_\kappa \in \Omega^1_\kappa(\mathrm{U}(1))$ with

$$z^m \cdot e_\kappa = e_\kappa \cdot \kappa^m z^m, \quad e_\kappa^2 = 0, \quad e_\kappa^* = e_\kappa, \quad \alpha_w(e_\kappa) \coloneqq e_\kappa;$$
$$\mathrm{d}(z^m) \coloneqq e_\kappa \cdot 2\pi \mathrm{i}[m]_\kappa z^m, \quad \mathrm{d}(e_k) \coloneqq 0.$$

Then $(\Omega_\kappa(\mathrm{U}(1)), \mathrm{d})$ is a $\mathrm{U}(1)$-invariant $*$-exterior algebra.

Remark

In fact, $(\Omega_\kappa(\mathrm{U}(1)), \mathrm{d})$ defines a complete $*$-calculus on $\mathcal{O}(\mathrm{U}(1))$.

Chevalley–Eilenberg extensions

Let (Ω, ∇) be a $\mathrm{U}(1)$-$*$-quasi-DGA; let $\kappa > 0$.

The *κ-deformed Chevalley–Eilenberg extension* of (Ω, ∇) is the $\mathrm{U}(1)$-quasi-DGA $(\mathrm{CE}_\kappa(\Omega), \mathrm{CE}_\kappa(\nabla))$, where:

1. $\mathrm{CE}_\kappa(\Omega)$ is obtained from Ω by adjoining $e_\kappa \in \mathrm{CE}_\kappa(\Omega)^1$ with

$$\omega \cdot e_\kappa = (-1)^{\deg(\omega)} e_\kappa \cdot \Lambda_\kappa(\omega), \qquad e_\kappa^2 = 0, \qquad e_\kappa^* = e_\kappa;$$

2. $\mathrm{CE}_\kappa(\nabla)$ is defined by

$$\mathrm{CE}_\kappa(\nabla)(\omega) \coloneqq e_\kappa \cdot \partial_\kappa(\omega) + \nabla(\omega), \quad \mathrm{CE}_\kappa(\nabla)(e_\kappa) \coloneqq 0;$$

3. the $\mathrm{U}(1)$-action is extended by defining e_κ to be $\mathrm{U}(1)$-invariant.

Differentiable U(1)-actions

Let (Ω, d) be a U(1)-$*$-exterior algebra; let $\kappa > 0$.

Then (Ω, d) is κ-*vertical* if $\mathsf{id}_{\Omega^0} : \Omega^0 \to \mathsf{CE}_\kappa(\Omega)^0$ extends to

$$\mathsf{ver} : (\Omega, \mathrm{d}) \to (\mathsf{CE}_\kappa(\Omega), \mathsf{CE}_\kappa(\mathrm{d})),$$

the *vertical coevaluation* on (Ω, d).

Remarks

1. In the commutative case, ver is contraction with $\left(\frac{\partial}{\partial t}\right)^{\#}$.
2. Being κ-vertical is completeness WRT $(\Omega_\kappa(\mathrm{U}(1)), \mathrm{d})$.
3. The corresponding vertical map à la Brzeziński–Majid is

$$\mathsf{ver}_{\mathrm{BM}} = (\mathsf{ver} - \mathsf{id})\big|_{\Omega^1} : \Omega^1 \to e_\kappa \cdot \Omega^0 \cong \mathbf{C}e_\kappa \otimes \Omega^0.$$

Vertical, horizontal, and basic forms

Let (Ω, d) be a κ-vertical U(1)-$*$-exterior algebra.

1. The U(1)-equivariant $*$-DGA of *vertical forms* is

$$(\Omega_{\mathrm{ver}}, \mathrm{d}_{\mathrm{ver}}) \coloneqq (\mathsf{CE}_\kappa(\Omega^0), \mathsf{CE}_\kappa(0)).$$

2. The U(1)-invariant graded $*$-sub-algebra of *horizontal forms* is

$$\Omega_{\mathrm{hor}} \coloneqq \ker(\mathrm{ver} - \mathrm{id}) = \{\omega \in \Omega \mid \mathrm{ver}\,\omega = \omega\}.$$

3. The trivially U(1)-equivariant $*$-DGA of *basic forms* is

$$(\Omega_{\mathrm{bas}}, \mathrm{d}_{\mathrm{bas}}) \coloneqq \left(\Omega_{\mathrm{hor}}^{\mathrm{U}(1)}, \mathrm{d}\big|_{\Omega_{\mathrm{hor}}^{\mathrm{U}(1)}}\right).$$

Differentiable quantum principal $\mathrm{U}(1)$-bundles

***Definition** (Brzeziński–Majid, Hajac, Đurđević, Beggs–Brzeziński, Beggs–Majid, Ć.)*

Given $\kappa > 0$, a κ-*differentiable quantum principal* $\mathrm{U}(1)$-*bundle* is a κ-vertical $\mathrm{U}(1)$-$*$-exterior algebra (Ω, d), such that:

1. there exist finite families $(e_i)_{i=1}^m$ and $(\epsilon_j)_{j=1}^n$ in $(\Omega^\circ)_1$, such that $\sum_{i=1}^m e_i e_i^* = 1 = \sum_{j=1}^n \epsilon_j^* \epsilon_k$;
2. Ω_{bas} is generated by $\Omega^\circ_{\mathrm{bas}}$ and $\mathrm{d}(\Omega^\circ_{\mathrm{bas}})$;
3. $\Omega_{\mathrm{hor}} = \Omega^\circ \cdot \Omega_{\mathrm{bas}} \cdot \Omega^\circ$.

Condition 1 implies that Ω° is a principal $\mathcal{O}(\mathrm{U}(1))$-comodule algebra and $(\Omega_{\mathrm{ver}}, \mathrm{d}_{\mathrm{ver}})$ is a $*$-exterior algebra.

Condition 2 implies that $(\Omega_{\mathrm{bas}}, \mathrm{d}_{\mathrm{bas}})$ is a $*$-exterior algebra.

Structure of horizontal forms

Let (Ω, d) be a κ-differentiable quantum principal $\mathrm{U}(1)$-bundle.

Proposition (Beggs–Majid, Cor. 5.53)

In fact, $\Omega_{\mathrm{hor}} = \Omega^{\circ} \cdot \Omega_{\mathrm{bas}}$.

Definition (Đurđević, Beggs–Majid, Ć.)

The *Fröhlich automorphism* of (Ω, d) is the unique $\mathrm{U}(1)$-equivariant automorphism Φ of $(Z\Omega_{\mathrm{bas}}, \mathrm{d}_{\mathrm{bas}})$, such that

$$\forall m \in \mathbf{Z}, \ \forall p \in (\Omega^{\circ})_m, \ \forall \beta \in Z\Omega_{\mathrm{bas}}, \qquad p \cdot \beta = \Phi(\beta) \cdot p.$$

Question

What is the relationship between the Fröhlich automorphism and the Đurđević braiding à la Del Donno–Latini–Weber?

Examples

1. Let $\theta \in \mathbf{R}$, and let $(\Omega_\theta(\mathbf{S}^3), \mathrm{d})$ be the θ-deformed de Rham calculus on $C^\infty_\theta(\mathbf{S}^3)$. Then $(\Omega_\theta(\mathbf{S}^3)^{\mathrm{alg}}, \mathrm{d})$ is a 1-differentiable quantum principal $\mathrm{U}(1)$-bundle with
$$(\Omega_\theta(\mathbf{S}^3)^{\mathrm{alg}}_{\mathrm{bas}}, \mathrm{d}_{\mathrm{bas}}) = (\Omega(\mathbf{C}\mathrm{P}^1), \mathrm{d})$$
and Φ given by rotation of $\mathbf{C}\mathrm{P}^1 \cong \mathbf{S}^2$ by $2\pi\theta$.
2. Let $q \in (0,1)$, let $(\Omega_q(\mathbf{S}^3), \mathrm{d})$ be the 3-dimensional calculus on $\mathcal{O}_q(\mathrm{SU}(2))$, and let $(\Omega_q(\mathbf{C}\mathrm{P}^1), \mathrm{d})$ be the 2-dimensional calculus on $\mathcal{O}_q(\mathbf{C}\mathrm{P}^1)$. Then $(\Omega_q(\mathbf{S}^3), \mathrm{d})$ is a q^2-differentiable quantum principal $\mathrm{U}(1)$ bundle with
$$(\Omega_q(\mathbf{S}^3)_{\mathrm{bas}}, \mathrm{d}_{\mathrm{bas}}) \cong (\Omega_q(\mathbf{C}\mathrm{P}^1), \mathrm{d})$$
and Φ uniquely determined by $\Phi(\mathrm{i}e^+e^-) = q^2\mathrm{i}e^+e^-$.

Principal connections

***Definition** (Brzeziński–Majid, Hajac, Đurđević, Beggs–Majid, Ć)*

Let (Ω, d) be a κ-differentiable quantum principal $\mathrm{U}(1)$-bundle.

1. A *connection* on (Ω, d) is a $\mathrm{U}(1)$-equivariant surjective $*$-homomorphism $\Pi : \Omega^\bullet \to \Omega^\bullet_{\mathrm{hor}}$, such that $\Pi^2 = \Pi$ and
$$\forall \omega \in \Omega^1, \qquad (\mathrm{id} - \Pi)(\omega)^2 = 0.$$
2. A *connection 1-form* on (Ω, d) is $\mathrm{U}(1)$-invariant self-adjoint $\vartheta \in \Omega^1$, such that
$$\alpha \cdot \vartheta = (-1)^{\deg(\alpha)} \vartheta \cdot \Lambda_\kappa(\alpha), \qquad \mathrm{ver}(\vartheta) = e_\kappa + \vartheta.$$

The set of connection 1-forms, if non-empty, is an affine space with space of translations $\{\omega \in Z\Omega^2_{\mathrm{bas}} \mid \omega^* = \omega\}$.

Principal connections

Proposition (Brzeziński–Majid, Đurđević, Ć.)

Let (Ω, d) *be a* κ*-differentiable quantum principal* $\mathrm{U}(1)$*-bundle. For every connection* Π*, there exists a unique connection 1-form* ϑ*, such that*

$$\forall p \in \Omega^{\circ}, \qquad (\mathrm{id} - \Pi) \circ \mathrm{d}(p) = \vartheta \cdot \partial_{\kappa}(p),$$

and vice versa. In that case, $\Omega^{\bullet} = \Omega^{\bullet}_{\mathrm{hor}} \oplus \vartheta_{\Pi} \cdot \Omega^{\bullet-1}_{\mathrm{hor}}$.

A connection Π restricts on Ω^1 to a $*$*-preserving strong bimodule connection* à la Brzeziński–Majid, Hajac, and Beggs–Majid.

A connection 1-form ϑ corresponds to a *multiplicative regular connection* à la Đurđević.

Curvature

Proposition (Đurđević, Ć.)

Let (Ω, d) *be a* κ*-differentiable quantum principal* $\mathrm{U}(1)$*-bundle; let* Φ *be its Fröhlich automorphism. Let* Π *be a connection on* (Ω, d)*; let* ϑ_Π *be its connection 1-form.*

1. *Let* $\mathcal{F}_\Pi := -\mathrm{d}(\vartheta_\Pi)$*. Then* $\mathcal{F}_\Pi$ *is closed, self-adjoint, basic, and central in* Ω_{bas}*, and* $\Phi(\mathcal{F}_\Pi) = \kappa\mathcal{F}_\Pi$*.*
2. *Let* $\mathrm{d}_{\mathrm{hor}} := \Pi \circ \mathrm{d}$*. Then* $(\Omega_{\mathrm{hor}}, \mathrm{d}_{\mathrm{hor}})$ *is a* $\mathrm{U}(1)$*-equivariant* $*$*-quasi-DGA, and* $\mathrm{d}_{\mathrm{hor}}^2(\omega) = \mathcal{F}_\Pi \cdot \partial_\kappa(\omega)$ *for* $\omega \in \Omega_{\mathrm{hor}}$*.*

We call $\mathcal{F}_\Pi$ the *curvature* of the connection Π on (Ω, d).

Example

The curvature of the *q-monopole* on $(\Omega_q(\mathbf{S}^3)_{\mathrm{bas}}, \mathrm{d}_{\mathrm{bas}})$ is $\frac{1}{2\pi}\mathrm{i}e^+e^-$.

Gysin sequence in de Rham cohomology

Theorem (Bouwknegt–Hanabuss–Mathai, Ć.)

Let (Ω, d) *be a* κ*-differentiable quantum principal bundle with connection* Π*. There is a long exact sequence*

$$\cdots \to H^k(\Omega_{\mathrm{bas}}) \xrightarrow{\cdot[\mathcal{F}_\Pi]} H^{k+2}(\Omega_{\mathrm{bas}}) \to H^{k+2}(\Omega) \xrightarrow{\int} H^{k+1}(\Omega_{\mathrm{bas}}) \to \cdots$$

where $\int[\omega_1 + \vartheta_\Pi \cdot \omega_2] := \left[\int_{\mathrm{U}(1)} \alpha_z(\omega_2)\,\mathrm{d}(z)\right]$.

Example

Use NC Hodge theory à la Prague on $(\Omega_q(\mathbf{C}\mathrm{P}^1), \mathrm{d})$ and Gysin with respect to the q-monopole to get an easy proof that

$$H^0(\Omega_q(\mathbf{S}^3)) = \mathbf{C}, \quad H^3(\Omega_q(\mathbf{S}^3)) = \mathbf{C}[e^0 e^+ e^-], \quad H^k(\Omega_q(\mathbf{S}^3)) = 0 \text{ else.}$$

Horizontal calculi

Let P be a *quantum topological principal* $\mathrm{U}(1)$*-bundle* with base B, i.e., P is a $\mathrm{U}(1)$-$*$-algebra admitting finite families $(e_i)_{i=1}^{m}$ and $(\epsilon_j)_{j=1}^{n}$ in $(\Omega^\circ)_1$, such that $\sum_{i=1}^{m} e_i e_i^* = 1 = \sum_{j=1}^{n} \epsilon_j^* \epsilon_k$.

Let (Ω_B, d_B) be a $*$-exterior algebra on $B \coloneqq P^{\mathrm{U}(1)}$ with $\Omega_B^\circ = B$.

Definition (Đurđević, Ć.)

A *horizontal calculus* for P with respect to (Ω_B, d_B) is a $\mathrm{U}(1)$-equivariant $*$-quasi-DGA $(\Omega_{P,\mathrm{hor}}, \mathrm{d}_{P,\mathrm{hor}})$, such that

$$\Omega_{P,\mathrm{hor}}^\circ = P, \quad (\Omega_{P,\mathrm{hor}}^{\mathrm{U}(1)}, \mathrm{d}_{P,\mathrm{hor}}\big|_{\Omega_{P,\mathrm{hor}}^{\mathrm{U}(1)}}) = (\Omega_B, \mathrm{d}_B),$$

$$\Omega_{P,\mathrm{hor}} = P \cdot \Omega_B \cdot P.$$

Associated line bundles

Theorem (Ć.)

1. *Hermitian line B-bimodules with Hermitian bimodule connections* WRT (Ω_B, d_B) *form a coherent 2-group* $\mathrm{DPic}(B)$ *on the nose.*
2. *The mapping* $m \mapsto (P_m, \mathrm{d}_{P,\mathrm{hor}}\big|_{P_m})$ *defines a homomorphism of coherent 2-groups* $\mathbf{Z} \to \mathrm{DPic}(B)$.
3. *For every object* $(\mathcal{L}, \nabla)$ *of* $\mathrm{DPic}(B)$, *there exists an essentially unique quantum topological principal* $\mathrm{U}(1)$*-bundle* P *with base* B *and horizontal calculus* $(\Omega_P, \mathrm{d}_{P,\mathrm{hor}})$ *for* P WRT (Ω_B, d_B), *such that*

$$(\mathcal{L}, \nabla) \cong (P_1, \mathrm{d}_{P,\mathrm{hor}}\big|_{P_1}).$$

In other words,

$$(\Omega_{P,\mathrm{hor}}, \mathrm{d}_{P,\mathrm{hor}}) \cong (\Omega_B, \mathrm{d}_B) \rtimes_{(\mathcal{L},\nabla)} \mathbf{Z}, \quad (\mathcal{L}, \nabla) \cong (P_1, \mathrm{d}_{P,\mathrm{hor}}\big|_{P_1}).$$

Curvature data

For convenience, let $\mathcal{S}(B) \coloneqq \{\omega \in Z\Omega_B^2 \mid \omega^* = \omega,\ \mathrm{d}_B(\omega) = 0\}$.

Definition (Đurđević, Ć.; cf. Beggs–Majid)

Let $(\Omega_{P,\mathrm{hor}}, \mathrm{d}_{P,\mathrm{hor}})$ be a horizontal calculus for P WRT (Ω_B, d_B).

1. Its *Fröhlich automorphism* is the unique $\mathrm{U}(1)$-equivariant automorphism Φ_P of $(Z\Omega_B, \mathrm{d}_B)$, such that

$$\forall m \in \mathbf{Z},\ \forall p \in P_m,\ \forall \beta \in Z\Omega_B, \qquad p \cdot \beta = \Phi_P(\beta) \cdot p.$$

2. Its *curvature 1-cocycle* is the unique 1-cocycle $\mathcal{F}_P : \mathbf{Z} \to \mathcal{S}(B)$ for the left $\mathbf{Z}$-action generated by $\Phi_P\big|_{\mathcal{S}(B)}$, such that

$$\forall m \in \mathbf{Z},\ \forall \omega \in (\Omega_{P,\mathrm{hor}})_m, \quad \mathrm{d}_{P,\mathrm{hor}}^2(\omega) = 2\pi\mathrm{i}\mathcal{F}_P(m) \cdot \omega.$$

Hence, its *curvature data* is $(\Phi_P, \mathcal{F}_P)$.

Curvature data

Example

Let (Ω, d) be a κ-differentiable quantum principal $U(1)$-bundle with connection Π.
Let $\mathrm{d}_{\mathrm{hor}} \coloneqq \Pi \circ \mathrm{d}\big|_{\Omega_{\mathrm{hor}}}$, so that $(\Omega_{\mathrm{hor}}, \mathrm{d}_{\mathrm{hor}})$ defines a horizontal calculus for Ω° with respect to $(\Omega_{\mathrm{bas}}, \mathrm{d}_{\mathrm{bas}})$.

1. The Fröhlich automorphism Φ_{Ω° of $(\Omega_{\mathrm{hor}}, \mathrm{d}_{\mathrm{hor}})$ is the Fröhlich automorphism Φ of (Ω, d).
2. The curvature 1-cocycle $\mathcal{F}_{\Omega^\circ}$ of $(\Omega_{\mathrm{hor}}, \mathrm{d}_{\mathrm{hor}})$ is given by

$$\mathcal{F}_{\Omega^\circ}(m) = [m]_\kappa \mathcal{F}_\Pi.$$

In particular, $\Phi_{\Omega^\circ}(\mathcal{F}_{\Omega^\circ}(1)) = \kappa \mathcal{F}_{\Omega^\circ}(1)$.

Vertical deformation parameter

Let $(\Omega_{P,\mathrm{hor}}, \mathrm{d}_{P,\mathrm{hor}})$ be a horizontal calculus for P WRT (Ω_B, d_B).

1. We have $\mathrm{d}_{P,\mathrm{hor}}^2 = 0$ IFF $\mathcal{F}_P = 0$, IFF $\mathcal{F}_P(1) = 0$; in this case, $(\Omega_{P,\mathrm{hor}}, \mathrm{d}_{P,\mathrm{hor}})$ is *flat*.
2. Suppose that $(\Omega_{P,\mathrm{hor}}, \mathrm{d}_{P,\mathrm{hor}})$ is not flat. Given $\kappa > 0$, we have $\mathrm{d}_{P,\mathrm{hor}}^2 = \mathcal{F}_P(1) \cdot \partial_\kappa(\cdot)$ IFF $\mathcal{F}_P = (m \mapsto [m]_\kappa \mathcal{F}_P(1))$, IFF

$$\Phi_P(\mathcal{F}_P(1)) = \kappa \mathcal{F}_P(1).$$

In this case, κ is the *vertical deformation parameter* of the horizontal calculus $(\Omega_{P,\mathrm{hor}}, \mathrm{d}_{P,\mathrm{hor}})$.

Vertical deformation parameter

Example

Let $\theta \in \mathbf{R} \setminus \mathbf{Q}$ be quadratic with square-free discriminant. Let $\epsilon_\theta = c\theta + d \in \mathbf{Z} + \mathbf{Z}\theta$ be the *norm-positive fundamental unit* of the real quadratic number field $\mathbf{Q}[\theta]$. Let $\mathcal{L}$ be the *basic Heisenberg module* on $C^\infty_\theta(\mathbf{T}^2)$ of rank ϵ_θ and degree c; let ∇ be its canonical constant curvature connection. Then $(\Omega_\theta(\mathbf{T}^2), \mathrm{d}) \rtimes_{(\mathcal{L},\nabla)} \mathbf{Z}$ has curvature data $(\Phi, \mathcal{F})$ defined by

$$\Phi = \bigoplus_{m=0}^{2} \epsilon_\theta^m \, \mathrm{id}_{\bigwedge \mathbf{R}^2}, \qquad \mathcal{F}(m) = [m]_{\epsilon_\theta^2} c e^1 e^2,$$

hence, vertical deformation parameter $\epsilon_\theta^2 > 1$.

Synthesis of total calculi

Theorem (Đurđević, Ć.)

Let $(\Omega_{P,\mathrm{hor}}, \mathrm{d}_{P,\mathrm{hor}})$ *be a horizontal calculus for* P *WRT* (Ω_B, d_B)*. Let* $\kappa > 0$ *be given, and suppose that* $(\Omega_{P,\mathrm{hor}}, \mathrm{d}_{P,\mathrm{hor}})$ *is flat or has deformation parameter* κ*. There exists essentially unique* κ*-differentiable quantum principal* $\mathrm{U}(1)$*-bundle with connection* $(\Omega_P, \mathrm{d}_P; \Pi_P)$*, such that*

$$\Omega_P^0 = P, \qquad ((\Omega_P)_{\mathrm{hor}}, (\mathrm{d}_P)_{\mathrm{hor}}) \cong (\Omega_{P,\mathrm{hor}}, \mathrm{d}_{P,\mathrm{hor}}).$$

Without loss of generality, $\Omega_P = \mathsf{CE}_\kappa(\Omega_{P,\mathrm{hor}})$*,* Π_P *is projection on* $\Omega_{P,\mathrm{hor}}$ *along* $e_\kappa \cdot \Omega_{P,\mathrm{hor}}$*, and*

$$\mathrm{d}_P(\omega) = e_\kappa \cdot \partial_\kappa(\omega) + \mathrm{d}_{P,\mathrm{hor}}(\omega), \qquad \mathrm{d}_P(e_k) = -\mathcal{F}_P(1).$$