ERRATUM: A RECONSTRUCTION THEOREM FOR ALMOST-COMMUTATIVE SPECTRAL TRIPLES

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Abstract. We correct the pre-orientability axiom in our modified definition of commutative spectral triple, and accordingly correct the argument extending the reconstruction theorem for (orientable) commutative spectral triples to the non-orientable (viz., only pre-orientable) case.

The pre-orientability axiom in [2, Def. 2.7] does not hold in general for Dirac-type operators on Clifford module bundles over odd-dimensional manifolds, since the chirality element, in that case need not act as the identity [4, §II.5; 5, §8]. This, however, can be readily corrected, and the proof of [2, Cor. 2.19] (and thus of the main result, [2, Thm. 2.17]) can then be modified to accommodate the corrected definition.

First, we correct the pre-orientability axiom on a commutative spectral triple $(\mathcal{A}, \mathcal{H}, D)$ of metric dimension p, to hold in the necessary generality:

- **Pre-orientability**: There exists an antisymmetric Hochschild p-cycle $c \in Z_p(\mathcal{A}, \mathcal{A})$ such that $\chi = \pi_D(c)$ is a self-adjoint unitary satisfying $a\chi = \chi a$ and $[D, a]\chi = (-1)^{p+1}\chi[D, a]$ for all $a \in \mathcal{A}$.

Given this, the definition of orientability must be reworded as follows:

Definition. A commutative spectral triple $(\mathcal{A}, \mathcal{H}, D)$ of metric dimension p is called *orientable* if p is even and $D\chi = -\chi D$, or if p is odd and $\chi = 1$.

Let us now turn to the proof of [2, Cor. 2.19], in the case of p odd. Suppose, then, that $(\mathcal{A}, \mathcal{H}, D)$ is a commutative spectral triple of metric dimension p, with p odd, let c be the antisymmetric Hochschild p-cycle given by pre-orientability, and let $\chi = \pi_D(c)$. Write $D = D_0 + M$ for

$$D_0 := \frac{1}{2}(D + \chi D\chi), \quad M := \frac{1}{2}(D - \chi D\chi).$$

Since χ commutes with elements of $[D, \mathcal{A}]$, M commutes with $\mathcal{A}$, and hence M is a self-adjoint element of $\operatorname{End}_{\mathcal{A}}(\mathcal{H}^\infty)$. The argument for the even case, *mutatis mutandis*, together with the following strengthened version of [2, Lem. A.10], then shows that $(\mathcal{A}, \mathcal{H}, D_0)$ is still a commutative spectral triple of metric dimension p, with $\pi_{D_0}(c) = \pi_D(c) = \chi$ and $D_0\chi = \chi D_0$:

Lemma. *If $(\mathcal{A}, \mathcal{H}, D)$ is strongly regular and of metric dimension $p \in \mathbb{N}$, and if $M \subset \operatorname{End}_{\mathcal{A}}(\mathcal{H}^\infty)$ for $\mathcal{H}^\infty = \cap_k \operatorname{Dom} D^k$, then for all $T \in B(\mathcal{H})$, $⨍ T\,|D|^{-p} = ⨍ T\,|D_M|^{-p}$.*

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Proof. The case of p even is handled by [2, Lem. A.10], so suppose instead that p is odd, so that $p+1$ is even; by [2, Lem. A.11], it suffices to show that the operator $(D_M^2+1)^{-p/2} - (D^2+1)^{-p/2}$ is trace-class.

First, by the proof of [2, Lem. A.10], *mutatis mutandis*, and in particular by setting $n = \frac{p+1}{2} \in \mathbb{N}$, and then $\epsilon = \frac{p}{p+1}(n - i + \frac{1}{2}) > 0$ when considering the term corresponding to $0 \leq i \leq n$, one finds that $(D_M^2+1)^{-p/2\alpha} - (D^2+1)^{-p/2\alpha} \in \mathcal{L}^\alpha(\mathcal{H})$, where $\alpha = \frac{p}{p+1}$ satisfies $0 < \alpha < 1$, and $\mathcal{L}^q(\mathcal{H})$ denotes the qth Schatten ideal in $B(\mathcal{H})$. Thus, $\left|(D_M^2+1)^{-p/2\alpha} - (D^2+1)^{-p/2\alpha}\right|^\alpha \in \mathcal{L}^1(\mathcal{H})$, so that by the BKS inequality [1, Thm. 1], $(D_M^2+1)^{-p/2} - (D^2+1)^{-p/2}$ is indeed trace-class. □

Now, since χ commutes with D_0 and with all elements of $\mathcal{A}$, $D_1 = \chi D_0$ is a self-adjoint operator on $\mathcal{H}$ satisfying $D_1^2 = D_0^2$ and $[D_1, a] = \chi[D_0, a]$ for all $a \in \mathcal{A}$. Because of this, all the axioms for a commutative spectral triple of metric dimension p immediately follow for $(\mathcal{A}, \mathcal{H}, D_1)$ except for pre-orientability, but even then, since p is odd, $\pi_{D_1}(c) = \chi^p \pi_{D_0}(c) = \chi^{p+1} = 1$, so that $(\mathcal{A}, \mathcal{H}, D_1)$ is, in fact, orientable. We may therefore apply the reconstruction theorem for commutative spectral triples [3, Thm. 11.3] to obtain a compact oriented Riemannian p-manifold X; the rest then follows exactly as in the even-dimensional case.

References

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