REVISITING THE DIFFERENTIAL TOPOLOGY OF NONCOMMUTATIVE TORI

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Shameless self-promotion

1. B. Ćć., *Geometric foundations for classical* U(1)*-gauge theory on noncommutative manifolds*, Commun. Math. Phys. 405 (2024), article no. 209.
2. B. Ćć. and Timmavajjula Venkata Karthik, *Maxwell's equations on noncommutative oriented Riemannian manifolds*, to appear.
3. B. Ćć., *Classification of* U(1)*-instantons on higher-dimensional noncommutative tori*, in preparation.

1. Review of noncommutative tori

Basic definitions

Fix $N \geq 2$ and $\theta \in \Lambda^2 \mathbf{R}^N$.

The *noncommutative N-torus* A_θ is the universal unital C*-algebra generated by unitaries $\{U_{\mathbf{m}}\}_{\mathbf{m} \in \mathbf{Z}^{*N}}$ satisfying

$$\forall \mathbf{m}, \mathbf{n} \in \mathbf{Z}^{*N}, \qquad U_{\mathbf{m}} \star_\theta U_{\mathbf{n}} = \exp(-\pi \mathrm{i} \langle \mathbf{m} \wedge \mathbf{n}, \theta \rangle) U_{\mathbf{m}+\mathbf{n}}.$$

The canonical action α of $\mathbf{T}^N$ on A_θ is defined by

$$\forall t \in \mathbf{T}^N, \forall \mathbf{m} \in \mathbf{Z}^{*N}, \qquad \alpha_t(U_{\mathbf{m}}) := \exp(2\pi \mathrm{i} \langle \mathbf{m}, t \rangle) U_{\mathbf{m}}.$$

The $\mathbf{T}^N$-invariant faithful trace τ on A_θ is defined by

$$\forall \mathbf{m} \in \mathbf{Z}^{*N}, \qquad \tau(U_{\mathbf{m}}) := \delta_{0,\mathbf{m}}.$$

Properties

Recall that θ is *nondegenerate* iff the alternating bicharacter $\exp \circ 2\pi i\langle \cdot, \theta \rangle : \Lambda^2 \mathbf{Z}^{*N} \to \mathrm{U}(1)$ is nondegenerate.

If θ is nondegenerate, then A_θ is central.

Theorem (many hands)

If θ is nondegenerate, then A_θ:

1. *is simple with unique trace* (Slawny),
2. *has cancellation* (Rieffel),
3. *has stable rank* 1 *and real rank* 0 (Blackadar–Kumjian–Rørdam), *and*
4. *is an A***T***-algebra* (Elliott–Evans, Q. Lin, Boca, Kishimoto, Phillips).

Elementary differential topology

The *smooth noncommutative N-torus* is the dense unital $*$-subalgebra of all $a \in A_\theta$ such that

$$(t \mapsto \alpha_t(a)) \in C^\infty(\mathbf{T}^N, A_\theta).$$

The canonical $*$-exterior algebra on A_θ^∞ is $(\Omega_\theta, \mathrm{d})$, where $\Omega_\theta := A_\theta^\infty \otimes \Lambda \mathbf{C}^{*N}$ and

$$\forall \mathbf{m} \in \mathbf{Z}^{*N},\ \forall \omega \in \Lambda \mathbf{C}^{*N}, \quad \mathrm{d}(U_{\mathbf{m}} \otimes \omega) := U_{\mathbf{m}} \otimes 2\pi \mathrm{i} \mathbf{m} \wedge \omega.$$

Theorem (Connes)

We have a Chern character $\mathrm{Ch} : K(A_\theta^\infty) \to \Lambda \mathbf{C}^{*N} \simeq H(\Omega_\theta, \mathrm{d})$ *explicitly computable in terms of* $(\Omega_\theta, \mathrm{d})$ *and* τ.

K-theory via differential topology

Call θ *totally irrational* if $\langle \cdot, e^{\theta} \rangle : \Lambda^{ev} \mathbf{Z}^{*N} \to \mathbf{R}$ is injective.

Theorem (Connes, Bost)

The map $A_\theta^\infty \hookrightarrow A_\theta$ induces a strong isomorphism in K-theory.

Theorem (Elliott)

The Chern character induces $\mu : K(A_\theta) \xrightarrow{\simeq} \Lambda \mathbf{Z}^{*N}$ such that

$$\mu([A_\theta]) = 1, \qquad \langle \mu(\cdot), e^{\theta} \rangle = \tau,$$

whence $\tau : K_0(A_\theta) \to \mathbf{R}$ is injective iff θ is totally irrational.

Theorem (Rieffel)

If θ is nondegenerate, then

$$\mu(K(A_\theta)^+) = \{\beta \in \Lambda \mathbf{Z}^{*N} : \langle \beta, e^{\theta} \rangle > 0\} \cup \{0\}.$$

2. Classification of symmetries

Homeomorphisms

Let $G_\theta \leq \mathrm{GL}^{\mathrm{ev}}(\Lambda \mathbf{Z}^N)$ consist of all γ such that $\gamma(1) = 1$ and

$$\gamma \cdot \mathrm{e}^\theta \in \mathbf{R}_{>0}\, \mathrm{e}^\theta.$$

If $N = 2$ and $\theta_{12} := \langle \mathrm{e}^1 \wedge \mathrm{e}^2, \theta \rangle \notin \mathbf{Q}$, then $G_\theta \simeq \mathrm{GL}(2; \mathbf{Z})$ acting trivially on $\mathbf{Z} \oplus \Lambda^2 \mathbf{Z}^{*2}$ and contragrediently on $\mathbf{Z}^{*2}$.

Theorem (Elliott–Rørdam)

If θ is nondegenerate, then we have a short exact sequence

$$1 \to \overline{\mathrm{Inn}}(A_\theta) \to \mathrm{Aut}(A_\theta) \to G_\theta \to 1,$$

where $\overline{\mathrm{Inn}}(A_\theta)$ is closed, connected, and topologically simple.

Diffeomorphisms when $N = 2$

Let $\mathrm{PU}(A_\theta^\infty)^0 \trianglelefteq \mathrm{PU}(A_\theta^\infty)$ consist of all $[u]$ such that $\mu([u])_{(1)} = 0$.

Theorem (Elliott)

If $N = 2$ and $\theta_{12} := \langle e^1 \wedge e^2, \theta \rangle$ is Diophantine, then

$$\mathrm{Aut}(A_\theta^\infty) \cong (\mathrm{PU}(A_\theta^\infty)^0 \rtimes \mathbf{T}^2) \rtimes \mathrm{SL}(2;\mathbf{Z}),$$

$$\mathrm{Out}(A_\theta^\infty) \cong \frac{\mathbf{R}^2}{\mathbf{Z}^2 + \imath(\mathbf{Z}^{*2})\theta} \rtimes \mathrm{SL}(2;\mathbf{Z}).$$

Corollary

If $N = 2$ and θ_{12} is Diophantine, then there exists $\gamma \in G_\theta$ that does not lift to $\mathrm{Aut}(A_\theta^\infty)$.

Diffeomorphisms when $N \geq 2$

Let $G_\theta^\infty \leq \mathrm{GL}(N; \mathbf{Z})$ consist of all g such that

$$b_g(\theta) := \Lambda^2 g(\theta) - \theta \in \Lambda^2 \mathbf{Z}^N;$$

it embeds canonically (but non-trivially) in G_θ.

If θ is totally irrational, then $G_\theta^\infty = \mathrm{SL}(2, \mathbf{Z})$ if $N = 2$ and $G_\theta^\infty = \{\pm I\}$ otherwise, whence $b_g(\theta) = 0$ for all $g \in G_\theta^\infty$.

Theorem (Cuntz–Elliott–Goodman–Jørgensen)

If θ is non-degenerate and Diophantine, then the surjection $\mathrm{Aut}(A_\theta) \twoheadrightarrow G_\theta$ restricts to a surjection $\mathrm{Aut}(A_\theta^\infty) \twoheadrightarrow G_\theta^\infty$.

Corollary

If θ is totally irrational and Diophantine, then there exists $\gamma \in G_\theta$ that does not lift to $\mathrm{Aut}(A_\theta^\infty)$.

The extended diffeomorphism group

Let B be a unital pre-C*-algebra with $*$-exterior algebra (Ω, d).

Definition (Ć.)

Its group of *extended diffeomorphisms* is

$$\mathrm{DAut}(B) := \{(\beta, \varphi) \in \Omega^1_{\mathrm{sa}} \rtimes \mathrm{Aut}(\Omega) : \mathrm{d} - \varphi \circ \mathrm{d} \circ \varphi^{-1} = \mathrm{i}[\beta, \cdot]\},$$

and its group of *outer* extended diffeomorphism is

$$\mathrm{DOut}(B) := \mathrm{DAut}(B)/\{(-\mathrm{i}\,\mathrm{d}u \cdot u^*, \mathrm{Ad}_u) : u \in U(B)\}.$$

Remark

By Bratteli–Elliott–Jørgensen, if θ is nondegenerate and Diophantine, then every $\varphi \in \mathrm{Aut}(A_\theta^\infty)$ lifts to $\mathrm{DAut}(A_\theta^\infty)$.

Diffeomorphisms revisited

Theorem (Ć.–Venkata Karthik)

If θ is nondegenerate, then we have short exact sequences

$$1 \to \mathbf{R}^{*N} \times (\mathrm{PU}(A_\theta^\infty)^0 \rtimes \mathbf{T}^N) \to \mathrm{DAut}(A_\theta^\infty) \to G_\theta^\infty \to 1,$$

$$1 \to \frac{\mathbf{R}^{*N} \oplus \mathbf{R}^N}{\exp_{\mathrm{SO}(N,N)}(\theta)(\mathbf{Z}^{*N} \oplus \mathbf{Z}^N)} \to \mathrm{DOut}(A_\theta^\infty) \to G_\theta^\infty \to 1,$$

which split if $b_g(\theta) \in 2\Lambda^2 \mathbf{Z}^N$ for all $g \in G_\theta^\infty$.

Corollary

If θ is totally irrational, then there exists $\gamma \in G_\theta$ that does not lift to $\mathrm{DAut}(A_\theta^\infty)$.

3. Classification of line bundles

Topological line bundles

Proposition (Iglesias-Lachaud)

Suppose that θ is totally irrational, and let

$$\mathbf{K}_\theta := \{r \in \langle \Lambda^{\mathrm{ev}} \mathbf{Q}^{*N}, \mathrm{e}^\theta \rangle : r \cdot \langle \Lambda^{\mathrm{ev}} \mathbf{Q}^{*N}, \mathrm{e}^\theta \rangle \subseteq \langle \Lambda^{\mathrm{ev}} \mathbf{Q}^{*N}, \mathrm{e}^\theta \rangle\},$$
$$\mathcal{O}_\theta := \{r \in \langle \Lambda^{\mathrm{ev}} \mathbf{Z}^{*N}, \mathrm{e}^\theta \rangle : r \cdot \langle \Lambda^{\mathrm{ev}} \mathbf{Z}^{*N}, \mathrm{e}^\theta \rangle \subseteq \langle \Lambda^{\mathrm{ev}} \mathbf{Z}^{*N}, \mathrm{e}^\theta \rangle\}.$$

Then $\mathbf{K}_\theta$ is a real number field such that $[\mathbf{K}_\theta : \mathbf{Q}] \mid 2^{N-1}$ and $\mathcal{O}_\theta$ is an order in $\mathbf{K}_\theta$.

Theorem (Nawata-Watatani)

If θ is totally irrational, then we have a short exact sequence

$$1 \to \mathrm{Out}(A_\theta) \to \mathrm{Pic}(A_\theta) \xrightarrow{\tau} \mathcal{O}_\theta^\times \cap \mathbf{R}_{>0} \to 1.$$

The differentiable Picard group

Let B be a unital pre-C*-algebra with $*$-exterior algebra (Ω, d).

Let $\mathrm{DPic}(B)$ denote the set of unitary equivalence classes of pairs (E, ∇), where:

1. E is a strong Morita self-equivalence of B, and
2. ∇ is a Hermitian bimodule connection on E.

Theorem (Ć.)

1. *The set* $\mathrm{DPic}(B)$ *forms a group with respect to* $\otimes_B$.
2. *We have a canonical surjection*

$$\mathrm{DAut}(B) \twoheadrightarrow \{[E, \nabla] \in \mathrm{DPic}(B) : [E] = [B] \text{ in } V(B)\}$$

that descends to an isomorphism on $\mathrm{DOut}(B)$.

Differentiable line bundles

Let $\mathrm{Spin}(\theta) \leq \mathrm{Spin}(N,N)$ consist of all x such that:

1. $\rho^*(x) \in \mathrm{GL}^{\mathrm{ev}}(\Lambda \mathbf{Z}^{*N})$, and
2. $\rho(x)\mathrm{e}^{\theta} = \chi(x)\mathrm{e}^{\theta}$ for some $\chi(x) \in \mathbf{R}_{>0}$.

Theorem (Ć.; cf. Schwarz, Rieffel–Schwarz, Li, Elliott–Li)

If θ is nondegenerate, then we have a short exact sequence

$$1 \to \frac{\mathbf{R}^{*N} \oplus \mathbf{R}^{N}}{\exp_{\mathrm{SO}(N,N)}(\theta)(\mathbf{Z}^{*N} \oplus \mathbf{Z}^{N})} \to \mathrm{DPic}(A_\theta^\infty) \xrightarrow{M} \mathrm{Spin}(\theta) \to 1,$$

where $\chi \circ M = \tau$.

Differentiable line bundles when $N = 2$

Lemma (Nawata–Watatani)

Suppose that $N = 2$ *and* $\theta_{12} := \langle e^1 \wedge e^2, \theta \rangle \notin \mathbf{Q}$.

1. *If* θ_{12} *is quadratic with discriminant* Δ, *then* $\mathbf{K}_\theta = \mathbf{Q}(\theta_{12})$ *and* $\mathcal{O}_\theta$ *its order of discriminant* Δ, *whence* $\mathcal{O}_\theta^\times \cap \mathbf{R}_{>0} \cong \mathbf{Z}$.
2. *Otherwise,* $\mathbf{K}_\theta = \mathbf{Q}$ *and* $\mathcal{O}_\theta = \mathbf{Z}$, *whence* $\mathcal{O}_\theta^\times \cap \mathbf{R}_{>0} = 1$.

Corollary (Ć.)

If $N = 2$ *and* $\theta_{12} \notin \mathbf{Q}$, *then*

$$\mathrm{DPic}(A_\theta^\infty) \simeq \mathrm{DOut}(A_\theta^\infty) \rtimes (\mathcal{O}_\theta^\times \cap \mathbf{R}_{>0}).$$

Differentiable vs topological line bundles

Corollary (Ć.)

Suppose $N > 2$, θ is totally irrational, and $[\mathbf{K}_\theta : \mathbf{Q}] = 2^{N-1}$. For every $r \in \mathcal{O}_\theta^\times \cap \mathbf{R}_{>0}$, if r lifts to $\mathrm{DPic}(A_\theta^\infty)$*, then $\mathcal{N}_{\mathbf{K}_\theta/\mathbf{Q}}(r) = 1$.*

Example

Choose $\theta \in \Lambda^2 \mathbf{R}^3$ such that $\mathbf{K}_\theta = \mathbf{Q}(\sqrt{5}, \sqrt{13})$ and $\mathcal{O}_\theta$ is its ring of integers, e.g.,

$$\theta := \tfrac{1+\sqrt{5}}{2} e_1 \wedge e_2 + \tfrac{1+\sqrt{13}}{2} e_1 \wedge e_3 + \tfrac{1+\sqrt{5}+\sqrt{13}+\sqrt{65}}{4} e_2 \wedge e_3.$$

Then $\frac{9+\sqrt{5}+\sqrt{13}+\sqrt{65}}{4} \in \mathcal{O}_\theta^\times \cap \mathbf{R}_{>0}$ does not lift to $\mathrm{DPic}(A_\theta^\infty)$.

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